Curvature Tensor of the Stationary Accelerated Frame in Gravity Field

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We define an accelerated frame that moves along $\hat{r}$-axis in the general relativistic curved space-time. We then calculate the curvature tensor of this accelerated frame in the stationary gravity field. The curvature tensor is divided into two parts: the curvature tensor as observed by the observer and the curvature tensor of the observer's own planet in the gravity field.

1. Introduction

The objective at first is to define an accelerated frame that moves along $\hat{r}$-axis in the curved space-time. In this context, the Schwarzschild solution is given as

$$d\tau^2 = (1-\frac{2GM}{rc^2})dt^2 - \frac{1}{c^2}[\frac{dr^2}{1-\frac{2GM}{rc^2}} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2] \quad (1)$$

Now, the acceleration of moving matter is a in the Schwarzschild space-time, which is given as

$$a = \frac{d}{dt}(\frac{u}{\sqrt{1-\frac{2GM}{rc^2}-\frac{u^2}{c^2}}}) = a_{\text{inertial}} - g$$

Here, a_{inertial} is the inertial acceleration, g is the pure gravity acceleration and u is given as

$$u = \frac{1}{\sqrt{1-\frac{2GM}{rc^2}}}\frac{dr}{dt} \quad (2)$$

If $a_0 = a/\sqrt{1-\frac{2GM}{rc^2}}$ then

$$a_0 = \frac{1}{\sqrt{1-\frac{2GM}{rc^2}}}\frac{d}{dt}(\frac{u}{\sqrt{1-\frac{2GM}{rc^2}-\frac{u^2}{c^2}}}) = \frac{d}{d\hat{t}}(\frac{V}{\sqrt{1-\frac{V^2}{c^2}}})$$

$$V = \frac{d\hat{r}}{d\hat{t}} = \frac{dr}{dt}\frac{1}{(1-\frac{2GM}{rc^2})}, \quad d\hat{t} = dt\sqrt{1-\frac{2GM}{rc^2}},$$

$$d\hat{r} = \frac{dr}{\sqrt{1-\frac{2GM}{rc^2}}}$$

$$a_0\hat{t} = \frac{V}{\sqrt{1-\frac{V^2}{c^2}}}, \quad V = \frac{a_0\hat{t}}{\sqrt{1+\frac{a_0^2\hat{t}^2}{c^2}}}, \quad (3)$$

Here, V is the velocity of $\hat{r}$-axis velocity

If, $\frac{d\theta}{dt} = \frac{d\phi}{dt} = 0$, the solution is

$$d\tau^2 = (1-\frac{2GM}{rc^2})dt^2 - \frac{1}{c^2}\frac{dr^2}{1-\frac{2GM}{rc^2}} = d\hat{t}^2 - \frac{1}{c^2}d\hat{r}^2 = d\hat{t}^2(1-\frac{V^2}{c^2}) \quad (4)$$

Now,

$$\tau = \int d\tau = \int\frac{d\hat{t}}{\sqrt{1+\frac{a_0^2\hat{t}^2}{c^2}}} = \frac{c}{a_0}\sinh^{-1}(\frac{a_0}{c}\hat{t}),$$

$$\hat{t} = \frac{c}{a_0}\sinh(\frac{a_0\tau}{c}),$$

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$$\hat{r}=\int V d\hat{t}=\int \frac{a_0 \hat{t} d\hat{t}}{\sqrt{1+\frac{a_0{}^2 \hat{t}^2}{c^2}}}=\frac{c^2}{a_0}\sqrt{1+\frac{a_0{}^2\hat{t}^2}{c^2}}$$
$$=\frac{c^2}{a_0}\cosh(\frac{a_0\tau}{c})$$
$$\frac{d\hat{t}}{d\tau}=\cosh(\frac{a_0}{c}\tau)\ ,\ \ \frac{1}{c}\frac{d\hat{r}}{d\tau}=\sinh(\frac{a_0}{c}\tau) \qquad (5)$$

2. The Tetrad in Curved Space-time

The tetrad $e^{\hat{\alpha}}{}_{\hat{\mu}}$ is the unit vector, which is defined by the following formula

$$\eta_{\hat{\alpha}\hat{\beta}} e^{\hat{\alpha}}{}_{\hat{\mu}} e^{\hat{\beta}}{}_{\hat{\upsilon}} = g_{\hat{\mu}\hat{\upsilon}} \qquad (6)$$

Now, if the matter moves along $\hat{r}$-axis in the curved space-time

$$\eta_{\hat{\alpha}\hat{\beta}} e^{\hat{\alpha}}{}_{\hat{\mu}}(\tau) e^{\hat{\beta}}{}_{\hat{\upsilon}}(\tau) = g_{\hat{\mu}\hat{\upsilon}} = \eta_{\hat{\mu}\hat{\upsilon}}\ ,\ \ g_{\hat{\mu}\hat{\upsilon}} = \eta_{\hat{\mu}\hat{\upsilon}} \qquad (7)$$

Hence, Eqn. (6), Eqn. (7) become

$$\eta_{\hat{\alpha}\hat{\beta}} e^{\hat{\alpha}}{}_{\hat{0}}(\tau) e^{\hat{\beta}}{}_{\hat{0}}(\tau) = \eta_{\hat{0}\hat{0}} = -1 \qquad (8)$$

$$d\tau^2 = -\frac{1}{c^2}\eta_{\hat{\alpha}\hat{\beta}} d\hat{x}^{\alpha} d\hat{x}^{\beta}$$
$$\rightarrow\ \ -1 = \eta_{\hat{\alpha}\hat{\beta}}(\frac{1}{c}\frac{d\hat{x}^{\alpha}}{d\tau})(\frac{1}{c}\frac{d\hat{x}^{\beta}}{d\tau}) = \eta_{\hat{\alpha}\hat{\beta}} e^{\hat{\alpha}}{}_{\hat{0}}(\tau) e^{\hat{\beta}}{}_{\hat{0}}(\tau)$$

$$\hat{x}^{\alpha} = (c\hat{t}, \hat{r}, \hat{\theta}, \hat{\phi}) \qquad (9)$$

According to Eqn. (5) and Eqn. (9),

$$e^{\hat{\alpha}}{}_{\hat{0}}(\tau) = \frac{1}{c}\frac{d\hat{x}^{\alpha}}{d\tau} = (\cosh(\frac{a_0\tau}{c}), \sinh(\frac{a_0\tau}{c}), 0, 0) \qquad (10)$$

About $\hat{\theta}$-axis and $\hat{\phi}$-axis orientations, as

$$\eta_{\hat{2}\hat{2}} e^{\hat{2}}{}_{\hat{2}}(\tau) e^{\hat{2}}{}_{\hat{2}}(\tau) = \eta_{\hat{2}\hat{2}} = 1, \quad e^{\hat{\alpha}}{}_{\hat{2}}(\tau) = (0,0,1,0)$$
$$\eta_{\hat{3}\hat{3}} e^{\hat{3}}{}_{\hat{3}}(\tau) e^{\hat{3}}{}_{\hat{3}}(\tau) = \eta_{\hat{3}\hat{3}} = 1, \quad e^{\hat{\alpha}}{}_{\hat{3}}(\tau) = (0,0,0,1) \qquad (11)$$

The other vector, $e^{\hat{\alpha}}{}_{\hat{1}}(\tau)$ has to satisfy the tetrad condition given by Eqn. (6) and Eqn. (7), and therefore we have

$$e^{\hat{\alpha}}{}_{\hat{1}}(\tau) = (\sinh(\frac{a_0\tau}{c}), \cosh(\frac{a_0\tau}{c}), 0, 0) \qquad (12)$$

Now, since

$$\bar{e}_{\hat{t}}{}^{\rho} = (1/\sqrt{1-\frac{2GM}{rc^2}}, 0,0,0)$$
$$\bar{e}_{\hat{r}}{}^{\rho} = (0, \sqrt{1-\frac{2GM}{rc^2}}, 0,0)$$
$$\bar{e}_{\hat{\theta}}{}^{\rho} = (0,0,1/r,0)\ \ ,\ \ \bar{e}_{\hat{\phi}}{}^{\rho} = (0,0,0,1/r\sin\theta)$$
$$g_{\rho\sigma}\bar{e}_{\hat{\alpha}}{}^{\rho}\bar{e}_{\hat{\beta}}{}^{\sigma} = \eta_{\hat{\alpha}\hat{\beta}} \qquad (13)$$

$$\frac{a_0}{c}\hat{t} = \sinh(\frac{a_0}{c}\tau) = \frac{v/c}{\sqrt{1-v^2/c^2}}\ ,$$
$$\sqrt{1+\frac{a_0{}^2\hat{t}^2}{c^2}} = \cosh(\frac{a_0}{c}\tau) = \frac{1}{\sqrt{1-v^2/c^2}} \qquad (14)$$

Therefore, the Lorentz transformation $B^{\hat{\alpha}}{}_{\hat{\mu}}(v)$ is

$$B^{\hat{\alpha}}{}_{\hat{\mu}}(v) = \begin{pmatrix} \frac{1}{\sqrt{1-v^2/c^2}} & \frac{v/c}{\sqrt{1-v^2/c^2}} & 0 & 0 \\ \frac{v/c}{\sqrt{1-v^2/c^2}} & \frac{1}{\sqrt{1-v^2/c^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$= e^{\hat{\alpha}}{}_{\hat{\mu}}(\tau) = \begin{pmatrix} \cosh(\frac{a_0}{c}\tau) & \sinh(\frac{a_0}{c}\tau) & 0 & 0 \\ \sinh(\frac{a_0}{c}\tau) & \cosh(\frac{a_0}{c}\tau) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (15)$$

$$\bar{e}'_{\hat{\mu}}{}^{\rho} = B^{\hat{\alpha}}{}_{\hat{\mu}}(v)\bar{e}_{\hat{\alpha}}{}^{\rho} = e^{\hat{\alpha}}{}_{\hat{\mu}}(\tau)\bar{e}_{\hat{\alpha}}{}^{\rho} \qquad (16)$$

Hence,

$$g_{\rho\sigma}\bar{e}_{\hat{\alpha}}{}^{\rho}\bar{e}_{\hat{\beta}}{}^{\sigma} = \eta_{\hat{\alpha}\hat{\beta}}$$
$$g_{\rho\sigma}B^{\hat{\alpha}}{}_{\hat{\mu}}(v)\bar{e}_{\hat{\alpha}}{}^{\rho}B^{\hat{\beta}}{}_{\hat{\upsilon}}(v)\bar{e}_{\hat{\beta}}{}^{\sigma}$$
$$= g_{\rho\sigma}\bar{e}'_{\hat{\mu}}{}^{\rho}\,\bar{e}'_{\hat{\upsilon}}{}^{\sigma} = \eta_{\hat{\alpha}\hat{\beta}} e^{\hat{\alpha}}{}_{\hat{\mu}}(\tau) e^{\hat{\beta}}{}_{\hat{\upsilon}}(\tau) = \eta_{\hat{\mu}\hat{\upsilon}} \qquad (17)$$

3. The Accelerated Frame in the Curved Space-time

The accelerated frame $\hat{\xi}$ in the curved space-time can be defined as

$$d\tau^2 = -\frac{1}{c^2}\eta_{\hat{\alpha}\hat{\beta}} d\hat{x}^{\alpha} d\hat{x}^{\beta} = -\frac{1}{c^2}\eta_{\hat{\alpha}\hat{\beta}} \frac{\partial \hat{x}^{\alpha}}{\partial \hat{\xi}^{\mu}} \frac{\partial \hat{x}^{\beta}}{\partial \hat{\xi}^{\upsilon}} d\hat{\xi}^{\mu} d\hat{\xi}^{\upsilon}$$

$$= -\frac{1}{c^2}\eta_{\hat{\alpha}\hat{\beta}} e^{\hat{\alpha}}{}_{\hat{\mu}} e^{\hat{\beta}}{}_{\hat{\upsilon}} d\hat{\xi}^{\mu} d\hat{\xi}^{\upsilon}$$

$$= -\frac{1}{c^2} g_{\hat{\mu}\hat{\upsilon}} d\hat{\xi}^{\mu} d\hat{\xi}^{\upsilon} \quad (18)$$

$$e^{\hat{\alpha}}{}_{\hat{\mu}} = \frac{\partial \hat{x}^{\alpha}}{\partial \hat{\xi}^{\mu}} \quad , \quad \frac{\partial e^{\hat{\alpha}}{}_{\hat{0}}}{\partial \hat{\xi}^{1}} = \frac{\partial^2 \hat{x}^{\alpha}}{c\partial \hat{\xi}^{0} \partial \hat{\xi}^{1}} = \frac{\partial e^{\hat{\alpha}}{}_{\hat{1}}}{c\partial \hat{\xi}^{0}} \quad (19)$$

3.1. Case-1

Now, in Eqns. (10) - (12), if one uses $\hat{\xi}^0$ instead of τ , then one can obtain new equations as

$$e^{\hat{\alpha}}{}_{\hat{0}}(\hat{\xi}^0) = \frac{1}{c}\frac{\partial \hat{x}^{\alpha}}{\partial \hat{\xi}^{0}} = ((1+\frac{a_0\hat{\xi}^1}{c^2})\cosh(\frac{a_0\hat{\xi}^0}{c}), (1+\frac{a_0\hat{\xi}^1}{c^2})\sinh(\frac{a_0\hat{\xi}^0}{c}),0,0) \quad (20)$$

$$e^{\hat{\alpha}}{}_{\hat{1}}(\hat{\xi}^0) = \frac{\partial \hat{x}^{\alpha}}{\partial \hat{\xi}^{1}} = (\sinh(\frac{a_0\hat{\xi}^0}{c}), \cosh(\frac{a_0\hat{\xi}^0}{c}),0,0) \quad (21)$$

$$e^{\hat{\alpha}}{}_{\hat{2}}(\hat{\xi}^0) = \frac{\partial \hat{x}^{\alpha}}{\partial \hat{\xi}^{2}} = (0,0,1,0) \,, \quad e^{\hat{\alpha}}{}_{\hat{3}}(\hat{\xi}^0) = \frac{\partial \hat{x}^{\alpha}}{\partial \hat{\xi}^{3}} = (0,0,1,0) \quad (22)$$

$$d\hat{x}^{\alpha} = \frac{\partial \hat{x}^{\alpha}}{\partial \hat{\xi}^{\mu}} d\hat{\xi}^{\mu} = e^{\hat{\alpha}}{}_{\hat{0}}(\hat{\xi}^0) c d\hat{\xi}^0 + e^{\hat{\alpha}}{}_{\hat{1}}(\hat{\xi}^0) d\hat{\xi}^1 + e^{\hat{\alpha}}{}_{\hat{2}}(\hat{\xi}^0) d\hat{\xi}^2 + e^{\hat{\alpha}}{}_{\hat{3}}(\hat{\xi}^0) d\hat{\xi}^3 \quad (23)$$

Hence,

$$cd\hat{t} = cdt\sqrt{1-\frac{2GM}{rc^2}} = (1+\frac{a_0\hat{\xi}^1}{c^2})\cosh(\frac{a_0\hat{\xi}^0}{c}) c d\hat{\xi}^0 + \sinh(\frac{a_0\hat{\xi}^0}{c}) d\hat{\xi}^1$$

$$d\hat{r} = \frac{dr}{\sqrt{1-\frac{2GM}{rc^2}}} = (1+\frac{a_0\hat{\xi}^1}{c^2})\sinh(\frac{a_0\hat{\xi}^0}{c}) c d\hat{\xi}^0 + \cosh(\frac{a_0\hat{\xi}^0}{c}) d\hat{\xi}^1$$

$$d\hat{\theta} = d\hat{\xi}^2 \quad , \quad d\hat{\phi} = d\hat{\xi}^3 \quad (24)$$

$$d\tau^2 = (1-\frac{2GM}{rc^2})dt^2 - \frac{1}{c^2}[\frac{dr^2}{1-\frac{2GM}{rc^2}} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2]$$

$$= d\hat{t}^2 - \frac{1}{c^2}[d\hat{r}^2 + d\hat{\theta}^2 + d\hat{\phi}^2]$$

$$= (1+\frac{a_0\hat{\xi}^1}{c^2})^2 (d\hat{\xi}^0)^2 - \frac{1}{c^2}[(d\hat{\xi}^1)^2 + (d\hat{\xi}^2)^2 + (d\hat{\xi}^3)^2] \quad (25)$$

The co-ordinate transformation is

$$c\hat{t} = (\frac{c^2}{a_0} + \hat{\xi}^1)\sinh(\frac{a_0\hat{\xi}^0}{c}) \,,$$

$$\hat{r} = (\frac{c^2}{a_0} + \hat{\xi}^1)\cosh(\frac{a_0\hat{\xi}^0}{c}) - \frac{c^2}{a_0}$$

$$\hat{\theta} = \hat{\xi}^2 \,, \hat{\phi} = \hat{\xi}^3 \quad (26)$$

The inverse-transformation is

$$\hat{\xi}^0 = \frac{c}{a_0}\tanh^{-1}(\frac{c\hat{t}}{\hat{r} + \frac{c^2}{a_0}}) \,,$$

$$\hat{\xi}^1 = \sqrt{(\hat{r} + \frac{c^2}{a_0})^2 - c^2\hat{t}^2} - \frac{c^2}{a_0}$$

$$\hat{\xi}^2 = \hat{\theta} \,, \hat{\xi}^3 = \hat{\phi} \quad (27)$$

We now calculate the curvature tensor $R_{\hat{\mu}\hat{\upsilon}\hat{\rho}\hat{\lambda}}(\hat{\xi})$ thus obtaining the following expression

$$R_{\hat\mu\hat\upsilon\hat\rho\hat\lambda}(\hat\xi) = \frac{\partial\hat x^{\alpha}}{\partial\hat\xi^{\mu}}\frac{\partial\hat x^{\beta}}{\partial\hat\xi^{\upsilon}}\frac{\partial\hat x^{\gamma}}{\partial\hat\xi^{\rho}}\frac{\partial\hat x^{\delta}}{\partial\hat\xi^{\lambda}} R_{\hat\alpha\hat\beta\hat\gamma\hat\delta}(\hat X)$$
$$= e^{\hat\alpha}{}_{\hat\mu}(\hat\xi^0)e^{\hat\beta}{}_{\hat\upsilon}(\hat\xi^0)e^{\hat\gamma}{}_{\hat\rho}(\hat\xi^0)e^{\hat\delta}{}_{\hat\lambda}(\hat\xi^0)R_{\hat\alpha\hat\beta\hat\gamma\hat\delta}(\hat X) \tag{28}$$

$$R_{\hat t\hat r\hat t\hat r} = -R_{\hat t\hat r\hat r\hat t} = R_{\hat r\hat t\hat r\hat t} = -R_{\hat r\hat t\hat t\hat r} = \frac{2GM}{r^3c^2},$$
$$R_{\hat t\hat\theta\hat t\hat\theta} = -R_{\hat t\hat\theta\hat\theta\hat t} = R_{\hat\theta\hat t\hat\theta\hat t} = -R_{\hat\theta\hat t\hat t\hat\theta} = -\frac{GM}{r^3c^2} = R_{\hat t\hat\phi\hat t\hat\phi}$$
$$= -R_{\hat t\hat\phi\hat\phi\hat t} = R_{\hat\phi\hat t\hat\phi\hat t} = -R_{\hat\phi\hat t\hat t\hat\phi}$$
$$R_{\hat\theta\hat\phi\hat\theta\hat\phi} = -R_{\hat\theta\hat\phi\hat\phi\hat\theta} = R_{\hat\theta\hat\phi\hat\theta\hat\phi} = -R_{\hat\theta\hat\phi\hat\phi\hat\theta} = -\frac{2GM}{r^3c^2}$$
$$R_{\hat r\hat\theta\hat r\hat\theta} = -R_{\hat r\hat\theta\hat\theta\hat r} = R_{\hat\theta\hat r\hat\theta\hat r} = -R_{\hat\theta\hat r\hat r\hat\theta} = \frac{GM}{r^3c^2} = R_{\hat r\hat\phi\hat r\hat\phi}$$
$$= -R_{\hat r\hat\phi\hat\phi\hat r} = R_{\hat\phi\hat r\hat\phi\hat r} = -R_{\hat\phi\hat r\hat r\hat\phi} \tag{29}$$

Therefore,

$$e^{\hat\alpha}{}_{\hat 0}(\hat\xi^0) = ((1+\frac{a_0\hat\xi^1}{c^2})\cosh(\frac{a_0\hat\xi^0}{c}),$$
$$(1+\frac{a_0\hat\xi^1}{c^2})\sinh(\frac{a_0\hat\xi^0}{c}),0,0)$$
$$e^{\hat\alpha}{}_{\hat 1}(\hat\xi^0) = (\sinh(\frac{a_0\hat\xi^0}{c}),\cosh(\frac{a_0\hat\xi^0}{c}),0,0)$$

$$e^{\hat\alpha}{}_{\hat 2}(\hat\xi^0) = (0,0,1,0)\,,\; e^{\hat\alpha}{}_{\hat 3}(\hat\xi^0) = (0,0,1,0) \tag{30}$$

$$R_{\hat 0\hat 1\hat 0\hat 1}(\hat\xi) = \frac{2GM}{r^3c^2}(1+\frac{a_0\hat\xi^1}{c^2})^2,$$
$$R_{\hat 0\hat 2\hat 0\hat 2}(\hat\xi) = R_{\hat 0\hat 3\hat 0\hat 3}(\hat\xi) = -\frac{GM}{r^3c^2}(1+\frac{a_0\hat\xi^1}{c^2})^2$$
$$R_{\hat 2\hat 3\hat 2\hat 3}(\hat\xi) = -\frac{2GM}{r^3c^2},$$
$$R_{\hat 1\hat 2\hat 1\hat 2}(\hat\xi) = R_{\hat 1\hat 3\hat 1\hat 3}(\hat\xi) = \frac{GM}{r^3c^2} \tag{31}$$

Specially, when $t = 0$,

$$R_{\hat 0\hat 1\hat 0\hat 1}(\hat\xi) = \frac{2GM}{r^3c^2}(1+\frac{a_0\hat\xi^1}{c^2})^2 = \frac{2GM}{r^3c^2}(1+\frac{a_0\hat r}{c^2})^2$$
$$= \frac{2GM}{r^3c^2}[1+\frac{a_0}{c^2}\{\sqrt{r}\sqrt{r-\frac{2GM}{c^2}} + \frac{2GM}{c^2}\ln|\sqrt{r}+\sqrt{r-\frac{2GM}{c^2}}|$$
$$-\sqrt{r_0}\sqrt{r_0-\frac{2GM}{c^2}} - \frac{2GM}{c^2}\ln|\sqrt{r_0}+\sqrt{r_0-\frac{2GM}{c^2}}|\}]^2$$

$$u = \frac{1}{\sqrt{1-\frac{2GM}{rc^2}}}\frac{dr}{dt} = 0 \rightarrow$$
$$V = \frac{d\hat r}{d\hat t} = \frac{a_0\hat t}{\sqrt{1+\frac{{a_0}^2\hat t^2}{c^2}}} = \frac{dr}{dt}\frac{1}{(1-\frac{2GM}{rc^2})} = 0 \tag{32}$$

Therefore, if, $t = \hat t = \hat\xi^0 = 0$, then the theory treats the real situation.

$$\hat\xi^1 = \sqrt{(\hat r+\frac{c^2}{a_0})^2 - c^2\hat t^2} - \frac{c^2}{a_0} = \hat r$$
$$d\hat r = \frac{dr}{\sqrt{1-\frac{2GM}{rc^2}}} \quad \rightarrow$$
$$\hat r = \sqrt{r}\sqrt{r-\frac{2GM}{c^2}} + \frac{2GM}{c^2}\ln|\sqrt{r}+\sqrt{r-\frac{2GM}{c^2}}|$$
$$-\sqrt{r_0}\sqrt{r_0-\frac{2GM}{c^2}} - \frac{2GM}{c^2}\ln|\sqrt{r_0}+\sqrt{r_0-\frac{2GM}{c^2}}|$$
$$a_0 = \frac{1}{\sqrt{1-\frac{2GM}{rc^2}}}\frac{d}{dt}(\frac{u}{\sqrt{1-\frac{2GM}{rc^2}-\frac{u^2}{c^2}}}) = \frac{a}{\sqrt{1-\frac{2GM}{rc^2}}}$$

$$a = \frac{d}{dt}(\frac{u}{\sqrt{1-\frac{2GM}{rc^2}-\frac{u^2}{c^2}}}) \tag{33}$$

This is equal to – g, the pure gravity acceleration. In Eqn. (33), r_0 is the location of the stationary accelerated frame.

In the curved space-time, the curvature tensor $R_{\hat\mu\hat\upsilon\hat\rho\hat\lambda}(\hat\xi)$ of the stationary accelerated frame is

$$= \frac{2GM}{r^3c^2}[1+\frac{1}{c^2}\frac{a}{\sqrt{1-\frac{2GM}{rc^2}}}\{\sqrt{r}\sqrt{r-\frac{2GM}{c^2}}+\frac{2GM}{c^2}\ln|\sqrt{r}+\sqrt{r-\frac{2GM}{c^2}}|$$
$$-\sqrt{r_0}\sqrt{r_0-\frac{2GM}{c^2}}-\frac{2GM}{c^2}\ln|\sqrt{r_0}+\sqrt{r_0-\frac{2GM}{c^2}}|\}]^2$$

$$R_{\hat{0}\hat{2}\hat{0}\hat{2}}(\hat{\xi}) = R_{\hat{0}\hat{3}\hat{0}\hat{3}}(\hat{\xi}) = -\frac{GM}{r^3c^2}(1+\frac{a_0\hat{\xi}^1}{c^2})^2 = -\frac{GM}{r^3c^2}(1+\frac{a_0\hat{r}}{c^2})^2$$
$$= -\frac{GM}{r^3c^2}[1+\frac{a_0}{c^2}\{\sqrt{r}\sqrt{r-\frac{2GM}{c^2}}+\frac{2GM}{c^2}\ln|\sqrt{r}+\sqrt{r-\frac{2GM}{c^2}}|$$
$$-\sqrt{r_0}\sqrt{r_0-\frac{2GM}{c^2}}-\frac{2GM}{c^2}\ln|\sqrt{r_0}+\sqrt{r_0-\frac{2GM}{c^2}}|\}]^2$$
$$= -\frac{GM}{r^3c^2}[1+\frac{1}{c^2}\frac{a}{\sqrt{1-\frac{2GM}{rc^2}}}\{\sqrt{r}\sqrt{r-\frac{2GM}{c^2}}+\frac{2GM}{c^2}\ln|\sqrt{r}+\sqrt{r-\frac{2GM}{c^2}}|$$
$$-\sqrt{r_0}\sqrt{r_0-\frac{2GM}{c^2}}-\frac{2GM}{c^2}\ln|\sqrt{r_0}+\sqrt{r_0-\frac{2GM}{c^2}}|\}]^2$$

$$R_{\hat{2}\hat{3}\hat{2}\hat{3}}(\hat{\xi}) = -\frac{2GM}{r^3c^2}\,,\quad R_{\hat{1}\hat{2}\hat{1}\hat{2}}(\hat{\xi}) = R_{\hat{1}\hat{3}\hat{1}\hat{3}}(\hat{\xi}) = \frac{GM}{r^3c^2} \qquad (34)$$

3.2. Case-2

Now, in Eqns. (10), (11) and (12), if one uses $\hat{\xi}^0$ instead of τ and multiply by $\exp(\frac{a_0}{c^2}\hat{\xi}^1)$

$$e^{\hat{\alpha}}{}_{\hat{0}}(\hat{\xi}^0) = \frac{1}{c}\frac{\partial\hat{x}^\alpha}{\partial\hat{\xi}^0} = (\exp(\frac{a_0}{c^2}\hat{\xi}^1)\cosh(\frac{a_0\hat{\xi}^0}{c}),\ \exp(\frac{a_0}{c^2}\hat{\xi}^1)\sinh(\frac{a_0\hat{\xi}^0}{c}),0,0) \qquad (35)$$

$$e^{\hat{\alpha}}{}_{\hat{1}}(\hat{\xi}^0) = \frac{\partial\hat{x}^\alpha}{\partial\hat{\xi}^1} = (\exp(\frac{a_0}{c^2}\hat{\xi}^1)\sinh(\frac{a_0\hat{\xi}^0}{c}),\ \exp(\frac{a_0}{c^2}\hat{\xi}^1)\cosh(\frac{a_0\hat{\xi}^0}{c}),0,0) \qquad (36)$$

$$e^{\hat{\alpha}}{}_{\hat{2}}(\hat{\xi}^0) = \frac{\partial\hat{x}^\alpha}{\partial\hat{\xi}^2} = (0,0,1,0)\,,$$
$$e^{\hat{\alpha}}{}_{\hat{3}}(\hat{\xi}^0) = \frac{\partial\hat{x}^\alpha}{\partial\hat{\xi}^3} = (0,0,1,0) \qquad (37)$$

$$d\hat{x}^\alpha = \frac{\partial\hat{x}^\alpha}{\partial\hat{\xi}^\mu}d\hat{\xi}^\mu = e^{\hat{\alpha}}{}_{\hat{0}}(\hat{\xi}^0)cd\hat{\xi}^0 + e^{\hat{\alpha}}{}_{\hat{1}}(\hat{\xi}^0)d\hat{\xi}^1 + e^{\hat{\alpha}}{}_{\hat{2}}(\hat{\xi}^0)d\hat{\xi}^2 + e^{\hat{\alpha}}{}_{\hat{3}}(\hat{\xi}^0)d\hat{\xi}^3 \qquad (38)$$

Hence,

$$cd\hat{t} = cdt\sqrt{1-\frac{2GM}{rc^2}} = \exp(\frac{a_0}{c^2}\hat{\xi}^1)\cosh(\frac{a_0\hat{\xi}^0}{c})cd\hat{\xi}^0 + \exp(\frac{a_0}{c^2}\hat{\xi}^1)\sinh(\frac{a_0\hat{\xi}^0}{c})d\hat{\xi}^1$$

$$d\hat{r} = \frac{dr}{\sqrt{1-\frac{2GM}{rc^2}}} = \exp(\frac{a_0}{c^2}\hat{\xi}^1)\sinh(\frac{a_0\hat{\xi}^0}{c})cd\hat{\xi}^0 + \exp(\frac{a_0}{c^2}\hat{\xi}^1)\cosh(\frac{a_0\hat{\xi}^0}{c})d\hat{\xi}^1$$

$$d\hat{\theta} = d\hat{\xi}^2 \quad , \quad d\hat{\phi} = d\hat{\xi}^3 \qquad (39)$$

$$d\tau^2 = (1-\frac{2GM}{rc^2})dt^2 - \frac{1}{c^2}[\frac{dr^2}{1-\frac{2GM}{rc^2}} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2] = d\hat{t}^2 - \frac{1}{c^2}[d\hat{r}^2 + d\hat{\theta}^2 + d\hat{\phi}^2]$$
$$= \exp(2\frac{a_0\hat{\xi}^1}{c^2})(d\hat{\xi}^0)^2 - \frac{1}{c^2}[\exp(2\frac{a_0\hat{\xi}^1}{c^2})(d\hat{\xi}^1)^2 + (d\hat{\xi}^2)^2 + (d\hat{\xi}^3)^2] \quad (40)$$

The coordinate transformation is

$$c\hat{t} = \frac{c^2}{a_0}\exp(\frac{a_0}{c^2}\hat{\xi}^1)\sinh(\frac{a_0\hat{\xi}^0}{c}),$$
$$\hat{r} = \frac{c^2}{a_0}\exp(\frac{a_0}{c^2}\hat{\xi}^1)\cosh(\frac{a_0\hat{\xi}^0}{c}) - \frac{c^2}{a_0}$$
$$\hat{\theta} = \hat{\xi}^2, \hat{\phi} = \hat{\xi}^3 \quad (41)$$

The inverse-transformation is

$$\hat{\xi}^0 = \frac{c}{a_0}\tanh^{-1}(\frac{c\hat{t}}{\hat{r}+\frac{c^2}{a_0}}),$$
$$\hat{\xi}^1 = \frac{c^2}{a_0}\ln|\frac{a_0}{c^2}\sqrt{(\hat{r}+\frac{c^2}{a_0})^2 - c^2\hat{t}^2}|$$
$$\hat{\xi}^2 = \hat{\theta}, \hat{\xi}^3 = \hat{\phi} \quad (42)$$

If we calculate the curvature tensor $R_{\hat{\mu}\hat{\upsilon}\hat{\rho}\hat{\lambda}}(\hat{\xi})$ as

$$R_{\hat{\mu}\hat{\upsilon}\hat{\rho}\hat{\lambda}}(\hat{\xi}) = \frac{\partial\hat{x}^\alpha}{\partial\hat{\xi}^\mu}\frac{\partial\hat{x}^\beta}{\partial\hat{\xi}^\upsilon}\frac{\partial\hat{x}^\gamma}{\partial\hat{\xi}^\rho}\frac{\partial\hat{x}^\delta}{\partial\hat{\xi}^\lambda}R_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}(\hat{X}) = e^{\hat{\alpha}}{}_{\hat{\mu}}(\hat{\xi}^0)e^{\hat{\beta}}{}_{\hat{\upsilon}}(\hat{\xi}^0)e^{\hat{\gamma}}{}_{\hat{\rho}}(\hat{\xi}^0)e^{\hat{\delta}}{}_{\hat{\lambda}}(\hat{\xi}^0)R_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}(\hat{X}) \quad (43)$$

$$R_{\hat{t}\hat{r}\hat{t}\hat{r}} = -R_{\hat{t}\hat{r}\hat{r}\hat{t}} = R_{\hat{r}\hat{t}\hat{r}\hat{t}} = -R_{\hat{r}\hat{t}\hat{t}\hat{r}} = \frac{2GM}{r^3c^2},$$
$$R_{\hat{t}\hat{\theta}\hat{t}\hat{\theta}} = -R_{\hat{t}\hat{\theta}\hat{\theta}\hat{t}} = R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}} = -R_{\hat{\theta}\hat{t}\hat{t}\hat{\theta}} = -\frac{GM}{r^3c^2} = R_{\hat{t}\hat{\phi}\hat{t}\hat{\phi}} = -R_{\hat{t}\hat{\phi}\hat{\phi}\hat{t}} = R_{\hat{\phi}\hat{t}\hat{\phi}\hat{t}} = -R_{\hat{\phi}\hat{t}\hat{t}\hat{\phi}}$$
$$R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = -R_{\hat{\theta}\hat{\phi}\hat{\phi}\hat{\theta}} = R_{\hat{\phi}\hat{\theta}\hat{\phi}\hat{\theta}} = -R_{\hat{\phi}\hat{\theta}\hat{\theta}\hat{\phi}} = -\frac{2GM}{r^3c^2}$$
$$R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} = -R_{\hat{r}\hat{\theta}\hat{\theta}\hat{r}} = R_{\hat{\theta}\hat{r}\hat{\theta}\hat{r}} = -R_{\hat{\theta}\hat{r}\hat{r}\hat{\theta}} = \frac{GM}{r^3c^2} = R_{\hat{r}\hat{\phi}\hat{r}\hat{\phi}} = -R_{\hat{r}\hat{\phi}\hat{\phi}\hat{r}} = R_{\hat{\phi}\hat{r}\hat{\phi}\hat{r}} = -R_{\hat{\phi}\hat{r}\hat{r}\hat{\phi}} \quad (44)$$

Then,

$$e^{\hat{\alpha}}{}_{\hat{0}}(\hat{\xi}^0) = (\exp(\frac{a_0}{c^2}\hat{\xi}^1)\cosh(\frac{a_0\hat{\xi}^0}{c}), \exp(\frac{a_0}{c^2}\hat{\xi}^1)\sinh(\frac{a_0\hat{\xi}^0}{c}),0,0)$$
$$e^{\hat{\alpha}}{}_{\hat{1}}(\hat{\xi}^0) = (\exp(\frac{a_0}{c^2}\hat{\xi}^1)\sinh(\frac{a_0\hat{\xi}^0}{c}), \exp(\frac{a_0}{c^2}\hat{\xi}^1)\cosh(\frac{a_0\hat{\xi}^0}{c}),0,0)$$
$$e^{\hat{\alpha}}{}_{\hat{2}}(\hat{\xi}^0) = (0,0,1,0), e^{\hat{\alpha}}{}_{\hat{3}}(\hat{\xi}^0) = (0,0,1,0) \quad (45)$$

$$R_{\hat{0}\hat{1}\hat{0}\hat{1}}(\hat{\xi}) = \frac{2GM}{r^3c^2}\exp(4\frac{a_0\hat{\xi}^1}{c^2}),$$
$$R_{\hat{0}\hat{2}\hat{0}\hat{2}}(\hat{\xi}) = R_{\hat{0}\hat{3}\hat{0}\hat{3}}(\hat{\xi}) = -\frac{GM}{r^3c^2}\exp(2\frac{a_0\hat{\xi}^1}{c^2})$$
$$R_{\hat{2}\hat{3}\hat{2}\hat{3}}(\hat{\xi}) = -\frac{2GM}{r^3c^2},$$
$$R_{\hat{1}\hat{2}\hat{1}\hat{2}}(\hat{\xi}) = R_{\hat{1}\hat{3}\hat{1}\hat{3}}(\hat{\xi}) = \frac{GM}{r^3c^2}\exp(2\frac{a_0}{c^2}\hat{\xi}^1) \quad (46)$$

As a special case, when $t = 0$, we obtain

$$u = \frac{1}{\sqrt{1-\frac{2GM}{rc^2}}}\frac{dr}{dt} = 0 \rightarrow$$
$$V = \frac{d\hat{r}}{d\hat{t}} = \frac{a_0\hat{t}}{\sqrt{1+\frac{a_0{}^2\hat{t}^2}{c^2}}} = \frac{dr}{dt}\frac{1}{(1-\frac{2GM}{rc^2})} = 0 \quad (47)$$

Therefore, if, $t = \hat{t} = \hat{\xi}^0 = 0$, the theory treats the real situation.

$$\hat{\xi}^1 = \frac{c^2}{a_0}\ln\left|\frac{a_0}{c^2}\sqrt{(\hat{r}+\frac{c^2}{a_0})^2 - c^2\hat{t}^2}\right| = \frac{c^2}{a_0}\ln\left|(1+\frac{a_0}{c^2}\hat{r})\right|$$

$$\exp(\frac{a_0}{c^2}\hat{\xi}^1) = 1+\frac{a_0}{c^2}\hat{r}$$

$$d\hat{r} = \frac{dr}{\sqrt{1-\frac{2GM}{rc^2}}} \quad\rightarrow\quad \hat{r} = \sqrt{r}\sqrt{r-\frac{2GM}{c^2}} + \frac{2GM}{c^2}\ln\left|\sqrt{r}+\sqrt{r-\frac{2GM}{c^2}}\right|$$

$$-\sqrt{r_0}\sqrt{r_0-\frac{2GM}{c^2}} - \frac{2GM}{c^2}\ln\left|\sqrt{r_0}+\sqrt{r_0-\frac{2GM}{c^2}}\right|$$

$$a_0 = \frac{1}{\sqrt{1-\frac{2GM}{rc^2}}}\frac{d}{dt}\left(\frac{u}{\sqrt{1-\frac{2GM}{rc^2}-\frac{u^2}{c^2}}}\right) = \frac{a}{\sqrt{1-\frac{2GM}{rc^2}}}$$

$$a = \frac{d}{dt}\left(\frac{u}{\sqrt{1-\frac{2GM}{rc^2}-\frac{u^2}{c^2}}}\right) \tag{48}$$

Now, in the curved space-time, the curvature tensor $R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}}(\hat{\xi})$ of the stationary accelerated frame is

$$R_{\hat{0}\hat{1}\hat{0}\hat{1}}(\hat{\xi}) = \frac{2GM}{r^3c^2}\exp(4\frac{a_0\hat{\xi}^1}{c^2}) = \frac{2GM}{r^3c^2}(1+\frac{a_0}{c^2}\hat{r})^4$$

$$= \frac{2GM}{r^3c^2}[1+\frac{a_0}{c^2}\{\sqrt{r}\sqrt{r-\frac{2GM}{c^2}} + \frac{2GM}{c^2}\ln\left|\sqrt{r}+\sqrt{r-\frac{2GM}{c^2}}\right|$$

$$-\sqrt{r_0}\sqrt{r_0-\frac{2GM}{c^2}} - \frac{2GM}{c^2}\ln\left|\sqrt{r_0}+\sqrt{r_0-\frac{2GM}{c^2}}\right|\}]^4$$

$$= \frac{2GM}{r^3c^2}[1+\frac{1}{c^2}\frac{a}{\sqrt{1-\frac{2GM}{rc^2}}}\{\sqrt{r}\sqrt{r-\frac{2GM}{c^2}} + \frac{2GM}{c^2}\ln\left|\sqrt{r}+\sqrt{r-\frac{2GM}{c^2}}\right|$$

$$-\sqrt{r_0}\sqrt{r_0-\frac{2GM}{c^2}} - \frac{2GM}{c^2}\ln\left|\sqrt{r_0}+\sqrt{r_0-\frac{2GM}{c^2}}\right|\}]^4$$

$$R_{\hat{0}\hat{2}\hat{0}\hat{2}}(\hat{\xi}) = R_{\hat{0}\hat{3}\hat{0}\hat{3}}(\hat{\xi}) = -\frac{GM}{r^3c^2}\exp(2\frac{a_0\hat{\xi}^1}{c^2}) = -\frac{GM}{r^3c^2}(1+\frac{a_0\hat{r}}{c^2})^2$$

$$= -\frac{GM}{r^3c^2}[1+\frac{a_0}{c^2}\{\sqrt{r}\sqrt{r-\frac{2GM}{c^2}} + \frac{2GM}{c^2}\ln\left|\sqrt{r}+\sqrt{r-\frac{2GM}{c^2}}\right|$$

$$-\sqrt{r_0}\sqrt{r_0-\frac{2GM}{c^2}} - \frac{2GM}{c^2}\ln\left|\sqrt{r_0}+\sqrt{r_0-\frac{2GM}{c^2}}\right|\}]^2$$

$$= -\frac{GM}{r^3c^2}[1+\frac{1}{c^2}\frac{a}{\sqrt{1-\frac{2GM}{rc^2}}}\{\sqrt{r}\sqrt{r-\frac{2GM}{c^2}} + \frac{2GM}{c^2}\ln\left|\sqrt{r}+\sqrt{r-\frac{2GM}{c^2}}\right|$$

$$-\sqrt{r_0}\sqrt{r_0-\frac{2GM}{c^2}} - \frac{2GM}{c^2}\ln\left|\sqrt{r_0}+\sqrt{r_0-\frac{2GM}{c^2}}\right|\}]^2$$

$$
\begin{aligned}
&R_{\hat{2}\hat{3}\hat{2}\hat{3}}(\hat{\xi}) = -\frac{2GM}{r^3c^2}\,,\\
&R_{\hat{1}\hat{2}\hat{1}\hat{2}}(\hat{\xi}) = R_{\hat{1}\hat{3}\hat{1}\hat{3}}(\hat{\xi}) = \frac{GM}{r^3c^2}\exp(2\frac{a_0}{c^2}\hat{\xi}^1) = \frac{GM}{r^3c^2}(1+\frac{a_0}{c^2}\hat{r})^2\\
&\quad = \frac{GM}{r^3c^2}[1+\frac{a_0}{c^2}\{\sqrt{r}\sqrt{r-\frac{2GM}{c^2}} + \frac{2GM}{c^2}\ln|\sqrt{r}+\sqrt{r-\frac{2GM}{c^2}}|\\
&\quad -\sqrt{r_0}\sqrt{r_0-\frac{2GM}{c^2}} - \frac{2GM}{c^2}\ln|\sqrt{r_0}+\sqrt{r_0-\frac{2GM}{c^2}}|\}]^2\\
&\quad = \frac{GM}{r^3c^2}[1+\frac{1}{c^2}\frac{a}{\sqrt{1-\frac{2GM}{rc^2}}}\{\sqrt{r}\sqrt{r-\frac{2GM}{c^2}} + \frac{2GM}{c^2}\ln|\sqrt{r}+\sqrt{r-\frac{2GM}{c^2}}|\\
&\quad -\sqrt{r_0}\sqrt{r_0-\frac{2GM}{c^2}} - \frac{2GM}{c^2}\ln|\sqrt{r_0}+\sqrt{r_0-\frac{2GM}{c^2}}|\}]^2
\end{aligned}
\tag{49}
$$

4. Conclusion

In the general relativity theory, we define the accelerated frame that moves in $\hat{r}$-axis in the curved time-space. Specially, if, $t = \hat{t} = \hat{\xi}^0 = 0$, this theory treats the curvature tensor of the stationary accelerated frame in the curved space-time in two-cases. In this context, the curvature tensor is divided into two parts: $R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}}(\hat{\xi})$ is the curvature tensor as observed by the observer and $R_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}(\hat{X})$ is the curvature tensor of the observer's own planet in the gravity field.

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