# NMR/MRI Blood Flow Magnetization Equation
# in the Rotating Frame of Reference: Part I

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This paper describes thoroughly the need and the method of deriving, the first of its kind, the NMR/MRI blood flow magnetization (*y*-component) equation in the rotating frame when rf $B_1$ field is applied along the laboratory X direction. The equation describing the $M_y$ component of magnetization in the rotating frame of reference (at resonance condition) in presence of flow of spins and relaxation times is given by:

$$\left[\left(v \bullet \nabla + \frac{\partial}{\partial t}\right)\frac{1}{\gamma B_1(x,t)}\left(v \bullet \nabla + \frac{\partial}{\partial t} + \frac{1}{T_2}\right) + \frac{1}{\gamma B_1(x,t)T_1}\left(v \bullet \nabla + \frac{\partial}{\partial t} + \frac{1}{T_2} + \gamma^2 B_1^2(x,t)T_1\right)\right]M_y = \frac{M_o}{T_1}$$

Where

$$v \bullet \nabla = v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y}$$

$v_x$ and $v_y$ are the components of blood flow velocity along the x and y directions of the rotating frame in an NMR experiment. The equation is expected to serve as the mother equation for accurate non invasive blood flow quantification through all NMR/MRI experiments. The method can be used to derive similar equations for the x and z component of magnetizations in the rotating frame of reference. It is shown how Awojyogbe's equation of blood flow magnetization can be obtained from above equation under the assumption of constant rf $B_1$ field and $v_y = 0$. The method of deriving the equation can be applied to obtain single component magnetization equation applicable for diffusion MRI/NMR experiments MRI of spins in presence of relaxation times and flow (if necessary). The method can also be extended to derive the NMR/MRI spin flow magnetization equation in presence of relaxation times and spin diffusion (if necessary), magnetic field gradients etc. in the laboratory frame of reference. The derivation of the corresponding flow equation for longitudinal component of magnetization will be discussed in a separate paper.

## 1. Introduction

Magnetic Resonance Imaging has stirred a great deal of interest in the medical community. As the image qualities are improving and as the amount of information that they can offer has been appreciated, the modality has become the primary topic of discussion among imaging scientists, imaging physicians, hospital administrators and government officials responsible for funding health care in many countries around the world. The combining applications of NMR and MRI in medical studies in human – a vast complex and very promising prospect – have not been adequately explored and are being investigated by scientists, engineers, mathematicians and clinicians around the world. At the heart of all these efforts

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lie the Bloch NMR equations for the three components of Magnetizations, $M_x$, $M_y$ and $M_z$ of the nuclear spin. NMR/MRI offers non-invasive measurements of blood flow in humans. This is very important in many types of clinical diagnosis. Magnetizations of flowing spins under given rf excitations are dependent on flow velocity of spins apart from rf $B_1$ field, magnetizing field, $B_o$ and field gradients and rf frequency etc. This forms the basis of measuring blood flow rate (both velocity and volume flow rate) by NMR/MRI techniques.

Various techniques have been developed over the past two decades in quantifying blood flow velocity and rates. Some of these are discussed below. Magnetic resonance (MR) signals are usually generated in three ways [1-5]: Spin Echo (SE), Free Induction Decay (FID) and Gradient Echo Technique. The MRI sequences depend on the integration of these types of signals with

gradient sequences necessary for spatial encoding. Flowing spins, along an applied magnetic field gradient, can acquire phase shift in the MR signal compared to the static tissue signal. The phase shift is proportional to the strength, duration of the gradient and motion of spins. Phase contrast as a result of the phase shift enables us to achieve complete suppression of the static tissue and thus estimation of information on flowing spins (usually of blood). The final signal in a given time frame should however depend not only on the phase shift but also on the flow dependent magnetizations. De [6] and Odoh and De [7] reported theoretical investigation and computation of time dependent CW NMR Blood flow signal for estimation of blood flow parameters using a new approach for continuous wave nuclear magnetic resonance (CW NMR). It has been shown that CW NMR can be a very useful technique that can be used for quantification of blood flow rate if the mathematical complexities involved can be handled successfully. However, in the analysis it was assumed that $B_1$ field is such that $M_z$ component does not vary appreciably from $M_o$, the saturation magnetization at a given bias magnetic field. This is true only for low values of $B_1$ field.

The feasibility of measuring blood flow to the human retina using arterial spin labeling MRI, a quantitative, noninvasive tomographic technique was investigated by Maleki et al. [8]. Quantification of cerebral blood flow (CBF) is very essential to diagnose many of the neurological pathologies that affect neonates and small infants as well as adults. CBF measurements are often performed through applications of radioactive tracers or other invasive methods which in turn can affect the human system. Varela et al. [9] developed technique using phase contrast MRI that can easily be appended to a neonatal MRI examination to provide rapid, robust, and non-invasive estimates of mean CBF, thus providing a means to monitor developmental or pathology-related alterations in cerebral perfusion and the impact of different treatment courses. Even though key hemodynamic factors for flow quantification, including arterial transit delay and the apparent decay time of the signal, were estimated by repeated measurements with different arterial spin labeling timing, it is expected that flow quantification by NMR/MRI in any part (through proper slice selection technique) of the entire human body would improve greatly if the Bloch NMR/MRI flow magnetization equations are properly framed. Awojoyogbe and his group's

works (2000-2012) [10-20] in this direction are noteworthy, some of which are discussed below:

Awojoyogbe [10-12] derived the following flow magnetization equation

$$V^2M'' + V(T_1^{-1} + T_2^{-1})M' + (\gamma^2B_1^2(x.t) + T_1^{-1}T_2^{-1})M$$

$$= \gamma M_oB_1(x.t)\ T_1^{-1} \qquad (A)$$

Where, V is the flow speed of the nuclear spin along the rotating $x$ direction along which the rf $B_1$ field is applied, $T_1$ and $T_2$ are the relaxation times and $B_1(x,t)$ is the rf $B_1$ field. $\gamma$ is the gyromagnetic ratio of the nuclear spin. M' and M'' are the first and second order derivatives (with respect to x) of the y component of magnetization. $M_o$ is the saturation magnetization. Dada et al. [21] suggested application of the Boubaker-Turki polynomials to Magnetic Resonance (MR) blood flow imaging. Their suggestion was based on the comparison of the following second order differential equation (B):

$$4x(1 – x^2)y'' + P(x,n)y' + Q(x,n) = 2Q(x,n)Tn(x)$$

$$(B)$$

Where,
$$P(x,n) = -4x^2 + 2nx – 2n + 8$$

$$Q(x,n) = -4x^2n + 6n – n^2\ -32$$

with the flow magnetization Eqn. (A).

Even though the correctness of Eqn. (B) is not checked in the present work, obviously Eqn. (A) does not fit in to the form of Eqn. (B), the reason being that V, the velocity of flowing spins (which is the object of investigation by NMR/MRI), cannot be forced to assume any chosen functions of x, so as to conform to Eqn. (B). Thus application of Boubaker polynomial for solution of Eqn. (1) is thus highly questionable. Moreover, as shown later Eqn. (1) for NMR/MRI flow magnetization is true only in rotating frame of reference and has limited validity. It is valid when V is independent of x and t, or in other words for steady blood flow through uniform vessel cross section and V is the flow speed of the spin along the rotating x direction along which the rf $B_1$ field is applied. For the equation to be valid, other conditions that must hold are: rf $B_1$ field must be independent of x and t, which is hard to achieve in any MRI/NMR sequence. A number of article were published by Awojoyogbe [10-13] and Awojoyogbe and his group [15-20] discussed mathematical concept of the Bloch flow equations for general magnetic

resonance imaging and exploring new dimensions cardiovascular flow and motion, application of Bloch NMR flow equations, Bessel and spherical harmonic functions based on the following ideas. Since molecular motion of water is significantly affected by macromolecules, the variation in the relaxation times between tissues is attributed to the effect of macromolecular interaction. The movement of water molecules during diffusion-driven random displacement is restricted by compartmental boundaries and other molecular obstacles in such a way that the actual diffusion distance is reduced as compared to what is expected in unrestricted diffusion. NMR/MRI sequences of diffusion driven spins should yield information on restricted compartmental boundaries through which diffusion of blood occurs in case of sicknesses such as sickle cell anemia. This is based on the idea that NMR/MRI signals in such cases would significantly differ (specially because of different relaxation times) from that of blood spin diffusion through unrestricted boundaries (as in healthy patients).

Jain et al. [22] discussed MRI estimation of global brain oxygen consumption rate. Their method of flow estimation like many other techniques are based on change of phase angle of magnetisation of flowing spins with magnetic field gradient pulse and flow velocity using the equation $\Delta\varphi = \gamma v \Delta M_1$, where $M_1 = \gamma v \left[\int G(t) t dt\right]$. $\Delta M_1$ is the difference in the first moment $M_1$ between two interleaves under magnetic field gradient. This equation is an over simplification in the sense that flowing blood magnetisation is itself velocity dependent and the equation does not take care of that and the equation could only yield good result when the spatial separation between the two interleaves as well as the flow velocity are quite small. Because the NMR/MRI magnetization is flow dependent, assumption of linear dependence of phase difference is ideally questionable.

Scientifically actual quantification of information on diseased vessels depends on correct Bloch NMR/MRI magnetization equation including flow, diffusion and relaxation effects of the spins. If this is not cast accurately from strict physical point of view it is prone to yield inaccurate information when quantification is made through application of such equations on the measured NMR/MRI signals or images. Moreover, extraction of flow information from NMR/MRI signals/images without the application of correct flow magnetization equation is bound to be associated with errors. Therefore, we have undertaken examination of the correct form of Bloch NMR flow magnetizations equation (the mother equation) from which NMR/MRI signals can be computed under various schemes of Imaging sequences for both steady and pulsatile blood (nuclear spin) flow. In this examination, we limit ourselves to the rotating frame of reference. We also limit ourselves to the formulation of flow magnetizations equation without applied field gradient. We believe that the case of applied field gradients can be easily included following our scheme of formulations. The corresponding equations in each Laboratory reference frames will be discussed in a subsequent paper Part II. We discuss the application of our equations to diffusion MRI.

In the following formulations and discussions, we assume that the bias magnetic field ($B_o$) is applied along laboratory $Z_o$ axis (of the laboratory $X_oY_oZ_o$ frame of axes) which coincides with the z axis of the rotating xyz axes) as shown in Fig. 1. The rf $B_1$ field in the form $B_{10}\cos wt$ is applied along the laboratory $X_o$ axis by an excitor coil as shown in Fig. 2a. A time independent magnetic field of magnitude $B_{10}/2$ can be considered to be along the rotating x axis making an angle wt with the laboratory $X_o$ axis counter clockwise. The detector coil has plane perpendicular to the $Y_o$ axis of the laboratory frame (Fig. 2b). Therefore, we have undertaken examination of the correct form of Bloch NMR flow magnetizations equation (the mother equation) from which NMR/MRI signals can be computed under various schemes of imaging sequences for both steady and pulsatile blood(nuclear spin) flow. In this examination, we limit ourselves to the rotating frame of references. We also limit ourselves to the formulation of flow magnetizations equation without applied field gradient. We believe that the case of applied field gradients can be easily included following our scheme of formulations. The corresponding equations in Laboratory reference frames will be discussed in a subsequent paper Part II. We discuss the application of our equations to diffusion MRI.
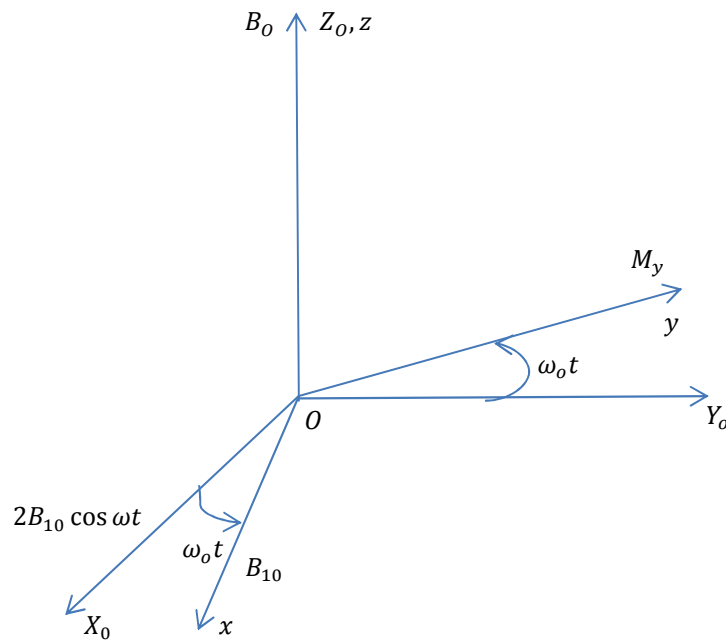
Fig.1: Rotational frame of axes (xyz) in relation to the fixed Laboratory frame of axes ($X_0Y_0Z_0$). x axis of the rotational frame makes an angle $\omega_0 t$ with the $X_0$ axis of the laboratory frame at time t. $\omega_0 = \gamma B_o = 2\pi f_o$. $f_o$ is the NMR resonance frequency. The oscillating rf $B_1$ field is applied along the $X_0$ axis along which the blood flow is considered to take place. Along the x axis of the rotating frame the rf $B_{10}$ field is independent of time. The magnetization component $M_y$ which is function of $B_{10}$, $T_1$, $T_2$, flow velocity etc. produces the signal in the detector coil placed in quadrature mode to the excitor coil (see Fig. 2).
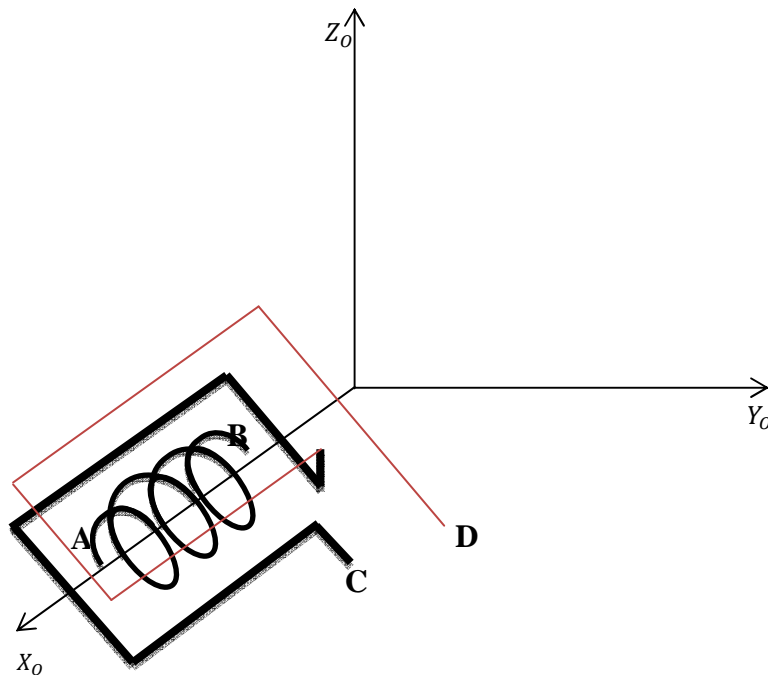


Fig.2: The relative disposition of the excitor coil (AB) and the detector coil (CD) in quadrature mode to be employed in the blood flow estimation by NMR.

## 2. Preliminary Basics of the Bloch NMR Equations

In NMR the nuclear spins are magnetized by an external magnetic field $B_o$ (along laboratory Z directions). Radio frequency (rf) excitation $B_1$ field are applied along laboratory X directions. Rf $B_1$ field of frequency $\omega$ is given by

$$\mathbf{B_1(t)} = \mathbf{B_{1o}}\sin(\omega t) \tag{1}$$

The net field is given by

$$\mathbf{B} = k_o B_o + i_o B_1(t) \tag{2}$$

as seen from the laboratory XYZ axes. Here, $i_o$ and $k_o$ are the unit vectors along X and Z axes. Please note that small letters x and y are used for rotating frame coordinates. The rotation of the nuclear spins under the net field can be described by the fact that the net rate of change of angular momentum $\left(\dfrac{dL}{dt}\right)$ is equal to the torque ($\tau = \mathbf{M} \times \mathbf{B}$) due to the net magnetic field.

Thus

$$\frac{dL}{dt} = \mathbf{M} \times \mathbf{B} \tag{3}$$

In some MRI sequences a pulsed magnetic field gradient may be applied along Z direction for selecting particular slices for investigation. We are omitting that from our present discussion. Now

$$\mathbf{M} = \gamma \mathbf{L} \tag{4}$$

Using Eqn. (4), Eqn. (3) becomes

$$\frac{dM}{dt} = \gamma \mathbf{M} \times \mathbf{B} \tag{5}$$

Eqn. (5) is correct only for a single nuclear spin in the absence of any interaction with neighbouring spins. But, when the spin is surrounded by the environment (such as water, tissues, fluids samples, and solids), the Z, X and Y components of the spin magnetization M relax due to the interaction with the environment. $M_z$ relaxes with time $T_1$ (called spin-lattice relaxation times) according to

$$M_z = M_{zo}(1 - \exp(-t/T_1)) \tag{6}$$

Where, $M_{zo}$ is the magnetization acquired by the sample as given by

$$M_{zo} = M_o B_J(\alpha) \tag{7}$$

$$M_o = N_A \mu_B (j(j+1))^{0.5} \tag{8a}$$

$\mu_B$ = Bohr magneton and $N_A$ = Avogradro number and where $B_J(\alpha)$ is the Brillouin function given by

$$B_J(\alpha) = \left( \frac{2j+1}{2j} \coth \frac{2j+1}{2j} \alpha - \frac{1}{2j} \coth \frac{1}{2j} \alpha \right) \tag{8b}$$

$$\alpha = B_o \mu_B (j(j+1))^{0.5}/k_B T \tag{8c}$$

and j is the total nuclear spin. $M_x$ and $M_y$ components relax with time $T_2$. With relaxation effects the Eqn. (5) then takes the form

$$\frac{dM_z}{dt} = \gamma(\mathbf{M} \times \mathbf{B})_\mathbf{z} \quad + \quad \frac{M_o - M_z}{T_1} \tag{9a}$$

$$\frac{dM_x}{dt} = \gamma(\mathbf{M} \times \mathbf{B})_\mathbf{x} \quad - \quad \frac{M_x}{T_2} \tag{10a}$$

$$\frac{dM_y}{dt} = \gamma(\mathbf{M} \times \mathbf{B})_\mathbf{y} \quad - \quad \frac{M_y}{T_2} \tag{11a}$$

Because of relations (Eqn. (1)) and (Eq. (2)) the solutions of (9a)-(11a) are difficult. A rotating frame of reference x,y,z is invoked where z axes of the rotating frame coincides with laboratory Z axis and the x-y axes rotate (with angular velocity $\omega$) in the laboratory X-Y plane anticlockwise such that the rotating x axis makes an angle $\omega t$ at any time t with the X axis. Then the rf $B_1$ field is along the rotating x axis in the rotating frame. Then Eqns. (9)-(11) become at NMR resonance conditions (i.e., at $\omega = \omega_o = \gamma B_o$).

$$\frac{dM_z}{dt} = \gamma(\mathbf{M} \times \mathbf{B_{1o}})_\mathbf{z} \quad + \quad \frac{M_o - M_z}{T_1} \tag{9b}$$

$$\frac{dM_x}{dt} = \gamma(\mathbf{M} \times \mathbf{B_{1o}})_\mathbf{x} \quad - \quad \frac{M_x}{T_2} \tag{10b}$$

$$\frac{dM_y}{dt} = \gamma(\mathbf{M} \times \mathbf{B_{1o}})_\mathbf{y} \quad - \quad \frac{M_y}{T_2} \tag{11b}$$

$$M = iM_x + jM_y + kM_z \tag{12}$$

Where, i,j and k are now unit vectors in the rotating frame of references as the components also are. In Eqns. (9b)-(11b) the rf $B_1$ field is independent of time in a CW NMR experiment and is dependent on time in a pulsed NMR experiment. Eqns. (9b)-(11b) then finally become ($\omega_o = \gamma B_o$).

$$\frac{dM_z}{dt} = -\gamma M_y B_{1o} + \frac{M_o - M_z}{T_1} \qquad (13)$$

$$\frac{dM_x}{dt} = -\frac{M_x}{T_2} \qquad (14)$$

$$\frac{dM_y}{dt} = \gamma M_z B_{1o} - \frac{M_y}{T_2} \qquad (15)$$

$B_{1o}$ is the amplitude of rf $B_1$ field. It is to be noted that the $M_y$ component produces the signals which are detected by coil placed usually at right angles to the rf $B_1$ field coil.

## 3. Formulations of NMR Flow Magnetization Equation

In this section, we derive the expression for the component of flow magnetization that will produce the final signal in the detector coil (with its axis along the laboratory X axis). Eqns. (13)-(15) are true in the rotating frames of references when there is no translational motion of the spins. Let us now consider the effect of translational motion (flow) of the nuclear spins on the above equations. With flow $M$ will become function of x,y,z and t, i.e.,

$$M = M(x,y,z,t) \qquad (16)$$

Then the total time derivative of $M$ is given by:

$$\frac{dM}{dt} = \frac{\partial M}{\partial t} + \frac{\partial M}{\partial x}\frac{dx}{dt} + \frac{\partial M}{\partial y}\frac{dy}{dt} + \frac{\partial M}{\partial z}\frac{dz}{dt} \qquad (17)$$

Where, $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ , are the components of the fluid velocity V.

We can also write this equation in the form:

$$\frac{dM}{dt} = \frac{\partial M}{\partial t} + \frac{\partial M}{\partial x}V_x + \frac{\partial M}{\partial y}V_y + \frac{\partial M}{\partial z}V_z \qquad (18)$$

and

$$\frac{dM}{dt} = \frac{\partial M}{\partial t} + V \cdot \nabla M \qquad (19)$$

$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$ is the gradient vector. The total derivative $\frac{dM}{dt}$ is also a function of *x, y, z,* and *t*. A similar relation holds between partial and total derivative of any quantity, and we may write, symbolically,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + V \cdot \nabla$$

Where, *V* is the fluid velocity.

Let us assume that the fluid is moving only along x direction. Then $V_y = V_z = 0$ and $V_x = V_x$. Then Eqn. (19) reduces to

$$\frac{dM}{dt} = \frac{\partial M}{\partial t} + V(\nabla M)_x = \frac{\partial M}{\partial t} + V\frac{\partial M}{\partial x} \qquad (20)$$

Now **M** is a vector quantity and is given by **M** = i$M_x$ + j$M_y$ + k$M_z$ .

Then from Eqn. (20) the Bloch flow NMR equations for flowing spins then [6,7] as follows.

$$\frac{dM_x}{dt} = \frac{\partial M_x}{\partial t} + V\frac{\partial M_x}{\partial x} = -\frac{M_x}{T_2} \qquad (21)$$

$$\frac{dM_y}{dt} = \frac{\partial M_y}{\partial t} + V\frac{\partial M_y}{\partial x} = \gamma M_z B_1(x) - \frac{M_y}{T_2} \qquad (22)$$

$$\frac{dM_z}{dt} = \frac{\partial M_z}{\partial t} + V\frac{\partial M_z}{\partial x} = -\gamma M_y B_1(x) + \frac{M_o - M_z}{T_1} \qquad (23)$$

Omitting the total derivative on the left of Eqns. (21)-(23)

$$\frac{\partial M_x}{\partial t} + V\frac{\partial M_x}{\partial x} = -\frac{M_x}{T_2} \qquad (21')$$

$$\frac{\partial M_y}{\partial t} + V\frac{\partial M_y}{\partial x} = \gamma M_z B_1(x) - \frac{M_y}{T_2} \qquad (22')$$

$$\frac{\partial M_z}{\partial t} + V\frac{\partial M_z}{\partial x} = -\gamma M_y B_1(x) + \frac{M_o - M_z}{T_1} \qquad (23')$$

The $M_y$ component in the above equations produces the signal in the detector coil when its cross section is placed normal to either the X or Y direction (note that the y axis makes ωt angle with the Y axis. Therefore, the task is how to solve for $M_y$ from Eqns. (22) and (23) above. In this paper, a new operator formalism is adopted to arrive at a differential equation involving $M_y$ only from the above equations. The above equations are true at NMR/MRI resonance conditions (i.e., the frequency, ω of the rf $B_1$ field is equal to $\gamma B_o$) in the rotating frame of reference (x,y,z) in which rotating frame $x$ axis coincides with the rotating $B_1(x)$ axis; $z$ axis coincides with the applied $B_o$ direction (LABORATORY Z axis) y axis is along $z$ x $x$ direction. $B_1$ field in a CW experiment is time independent in a CW NMR experiment. However, in pulsed NMR experiment $B_1$ field is time dependent even in a rotating frame. Hence forth $B_1(x)$ will be replaced by $B_1(x,t)$ in the following equations so as to represent a general case.

## 4.  Operator Formalism Method for Obtaining Bloch NMR Equations of Flow Magnetization, $M_y$ in the Rotating Frame of Reference

From the above equations, a partial differential equation of second order is derived following an operator formalism method which is very valuable in the analyses of space and time dependence of the NMR magnetization components of flowing spins. In Eqn. (21′) $M_x$ can be solved by assuming $M_x = f(x)g(t)$ and using separation of variable when $V$ is steady velocity or time dependent but independent of X. Below we use the symbol $v$ for the velocity $V$ in above equations. To solve for $M_y$ we see that From Eqn. (23′), we have

$$v\frac{\partial M_z}{\partial x}+\frac{\partial M_z}{\partial t}+\frac{M_z}{T_1}=-\gamma M_y B_1(x,t)+\frac{M_o}{T_1}$$

or

$$\left(v\frac{\partial}{\partial x}+\frac{\partial}{\partial t}+\frac{1}{T_1}\right)M_z=-\gamma M_y B_1(x,t)+\frac{M_o}{T_1}$$

or,

$$M_z=\frac{1}{\left(v\dfrac{\partial}{\partial x}+\dfrac{\partial}{\partial t}+\dfrac{1}{T_1}\right)}\left(-\gamma M_y B_1(x,t)+\frac{M_o}{T_1}\right)$$

$$(24)$$

In Eqn. (24) $\dfrac{1}{\left(v\dfrac{\partial}{\partial x}+\dfrac{\partial}{\partial t}+\dfrac{1}{T_1}\right)}$ is the inverse

operator of $\left(v\dfrac{\partial}{\partial x}+\dfrac{\partial}{\partial t}+\dfrac{1}{T_1}\right)$. Substituting $M_z$ from Eqn. (24) in Eqn. (22′), it follows that

$$\frac{\partial M_y}{\partial t}+v\frac{\partial M_y}{\partial x}+\frac{M_y}{T_2}$$

$$=\gamma\left\{\frac{1}{\left(v\dfrac{\partial}{\partial x}+\dfrac{\partial}{\partial t}+\dfrac{1}{T_1}\right)}\left(-\gamma M_y B_1(x,t)+\frac{M_o}{T_1}\right)\right\}B_1(x,t)$$

$$(25)$$

Or,

$$\left(v\frac{\partial}{\partial x}+\frac{\partial}{\partial t}+\frac{1}{T_2}\right)M_y$$

$$=\gamma\left\{\frac{1}{\left(v\dfrac{\partial}{\partial x}+\dfrac{\partial}{\partial t}+\dfrac{1}{T_1}\right)}\left(-\gamma M_y B_1(x,t)+\frac{M_o}{T_1}\right)\right\}B_1(x,t)$$

$$(26)$$

Eqn. (26) contains only $M_y$. In Eqn. (26) it is not easy to evaluate the effect of $\dfrac{1}{\left(v\dfrac{\partial}{\partial x}+\dfrac{\partial}{\partial t}+\dfrac{1}{T_1}\right)}$ on

the function $\left(-\gamma M_y B_1(x,t)+\dfrac{M_o}{T_1}\right)$.

We use the following method to eliminate the effect of the inverse operator and obtain a differential equation in $M_y$.

Let $\Omega_1$ and $\Omega_2$ represent the differential operators $\left(v\dfrac{\partial}{\partial x}+\dfrac{\partial}{\partial t}+\dfrac{1}{T_1}\right)$ and $\left(v\dfrac{\partial}{\partial x}+\dfrac{\partial}{\partial t}+\dfrac{1}{T_2}\right)$ and then

$\dfrac{1}{\left(v\dfrac{\partial}{\partial x}+\dfrac{\partial}{\partial t}+\dfrac{1}{T_1}\right)}$ be denoted by the symbol $\Omega_1^{-1}$

(i.e., inverse of $\Omega_1$. In Eqn. (25)) it is important to

note that $\Omega^{-1}$ acts only on $\left(-\gamma M_y B_1(x,t)+\dfrac{M_o}{T_1}\right)$ and not on $B_1(x,t)$ outside the curly bracket in Eqns. (25) and (26). Eqn. (26) is then written in terms of the operators as

$$\Omega_2 M_y = \gamma\left\{\Omega_1^{-1}\left(-\gamma M_y B_1(x)+\frac{M_o}{T_1}\right)\right\}B_1(x,t) \quad (27)$$

We note that the main problem in solving the Bloch NMR differential equations is handling the inverse operators of $\Omega_1$ and $\Omega_2$. We operate both sides of Eqn. (27) by $\Omega_1$. Then we get

$$\Omega_1[\Omega_2 M_y]$$
$$(28)$$
$$= \gamma\Omega_1\left[\left\{\Omega_1^{-1}\left(-\gamma M_y B_1(x,t)+\frac{M_o}{T_1}\right)\right\}B_1(x,t)\right]$$

In the right-hand side of Eqn. (28) we note that $\Omega_1$ acts also on $B_1(x,t)$ outside the curly bracket. Let

$$-\gamma M_y B_1(x,t)+\frac{M_o}{T_1} = f(x) \text{ and } \Omega_1^{-1}(f(x)) = g(x)$$
$$(29)$$

Then Eqn. (28) becomes

$$\Omega_1[\Omega_2 M_y] = \gamma\Omega_1[\{\Omega_1^{-1}(f(x))\}B_1(x,t)] \quad (30)$$

Or,

$$\Omega_1[\Omega_2 M_y] = \gamma\Omega_1[\{g(x)\}B_1(x,t)] \quad (30)$$

We now expand the right-hand side of Eqn. (30) first. We get after expanding

$$\Omega_1[\Omega_2 M_y]$$
$$= \gamma\left[B_1(x)\Omega_1\{g(x)\} + g(x)\left(\Omega_1 - \frac{1}{T_1}\right)B_1(x,t)\right] \quad (31)$$

Now we substitute the expression for *g(x)* in Eqn. (31). We get

$$\Omega_1[\Omega_2 M_y]$$
$$= \gamma\left[B_1(x,t)\Omega_1\{\Omega_1^{-1}f(x)\} + (\Omega_1^{-1}f(x))\left(\Omega_1 - \frac{1}{T_1}\right)B_1(x,t)\right]$$
$$(32)$$

$$\Omega_1[\Omega_2 M_y]$$
$$= \gamma\left[B_1(x,t)f(x) + \{\Omega_1^{-1}f(x)\}\left(\Omega_1 - \frac{1}{T_1}\right)B_1(x,t)\right]$$
$$(33)$$

In Eqn. (33) we have utilised the fact that $\Omega_1\Omega_1^{-1} = 1$.

Eqn. (27) is written in terms of f(x) as

$$\Omega_2 M_y = \gamma\{\Omega_1^{-1}f(x)\}B_1(x,t) \quad (27)$$

Eliminating $\{\Omega_1^{-1}f(x)\}$ from the Eqns. (33) and (27) we get

$$\Omega_1[\Omega_2 M_y]$$
$$= \gamma[B_1(x,t)f(x)]] + \{\Omega_2 M_y\}\left\{\left(\Omega_1 - \frac{1}{T_1}\right)B_1(x,t)\right\}/$$
$$B_1(x,t)$$
$$(34)$$

Substituting f(x) from Eqn. (29) into Eqn. (34) we get

$$\Omega_1[\Omega_2 M_y]$$
$$= \gamma\left[B_1(x,t)\left\{-\gamma M_y B_1(x,t)+\frac{M_o}{T_1}\right\}\right]$$
$$+ \{\Omega_2 M_y\}\left\{\left(\Omega_1 - \frac{1}{T_1}\right)B_1(x,t)\right\}/B_1(x,t) \quad (35)$$

Substituting the expressions for $\Omega_1,\Omega_2$ into Eqn. (35) we get (noting that the terms in the curly brackets of the second term of Eqn. (35) are multiplicative (and hence commutative).

$$\left(v\frac{\partial}{\partial x}+\frac{\partial}{\partial t}+\frac{1}{T_1}\right)\left(v\frac{\partial}{\partial x}+\frac{\partial}{\partial t}+\frac{1}{T_2}\right)M_y$$

$$=\gamma\left[B_1(x,t)\left\{-\gamma M_y B_1(x,t)+\frac{M_o}{T_1}\right\}\right]$$

$$+\left\{\left(v\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\right)B_1(x,t)\right\}\left\{\left(v\frac{\partial}{\partial x}+\frac{\partial}{\partial t}+\frac{1}{T_2}\right)M_y\right\}\bigg/$$

$$B_1(x,t) \qquad (36)$$

The differential Eqn. (36) can be written in a compact form as

$$\left[B_1(x,t)\left(v\frac{\partial}{\partial x}+\frac{\partial}{\partial t}+\frac{1}{T_1}\right)\left(v\frac{\partial}{\partial x}+\frac{\partial}{\partial t}+\frac{1}{T_2}\right)\right.$$

$$+\gamma^2 B_1^3(x,t)-\left\{\left(v\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\right)B_1(x,t)\right\}$$

$$\left.\left(v\frac{\partial}{\partial x}+\frac{\partial}{\partial t}+\frac{1}{T_2}\right)\right]M_y=\gamma\left[B_1^2(x,t)\frac{M_o}{T_1}\right] \qquad (37)$$

The operators appearing in Eqns. (36) and (37) are all commutative in the realm of NMR/MRI sequences, i.e., $v\dfrac{\partial}{\partial x}\dfrac{\partial}{\partial t}=\dfrac{\partial}{\partial t}\,v\dfrac{\partial}{\partial x}$, specially when $v$ corresponds to steady flow situation and independent of time. The solution of Eqn. (37) when $v$ is time dependent or pulsatile flow as in humans is out of the scope of this paper. Determination of $T_2$, $T_1$ relaxation times and MRI images are based on pulsed sequences when $B_1(x,t)$ is time dependent and usually given by a Fourier series in $\omega t$. We believe that the above Eqns. (36) and (37) should hold in such case also and the Eqn. (37) represents the true diffrential equation for $M_y$, which produces the detectable NMR signal under any NMR/MRI sequence.

## 5. Discussion

In absence of flow ($v = 0$), the equation of Bloch NMR magnetization $M_y$ is then given from Eqn. (37) as:

$$\left[B_1(x,t)\left(\frac{\partial}{\partial t}+\frac{1}{T_1}\right)\left(\frac{\partial}{\partial t}+\frac{1}{T_2}\right)+\gamma^2 B_1^3(x,t)-\right.$$

$$\left.\left\{\left(\frac{\partial}{\partial t}\right)B_1(x,t)\right\}\left(\frac{\partial}{\partial t}+\frac{1}{T_2}\right)\right]M_y=\gamma\left[B_1^2(x,t)\frac{M_o}{T_1}\right] \qquad (38)$$

Eqn. (38) can describe magnetization component $M_y$ of stationary spin in the rotating frame for all NMR/MRI pulse sequences in which $B_1(x,t)$ can be in the form of pulsed rf. For CW NMR (without flow) the explicit time derivative part can be dropped from Eqn. (38).

In such situation $B_1(x,t) = B_{10}$ and we see from Eqn. (38) that

$$\left(\frac{1}{T_1 T_2}+\gamma^2 B_1^{\ 2}\right)M_y=\frac{\gamma B_{1o}M_o}{T_1}$$

$$M_y=\frac{\gamma B_{1o}T_2 M_o}{1+\gamma^2 B_1^{\ 2}T_1 T_2} \qquad (39)$$

Where, $B_{1o}$ is the magnitude of rf $B_1$ field along the x axis of the rotating frame of reference. This makes an angle $\omega t$ with the laboratory $X_o$ axis. We note that the solution of Eqn. (21) for $M_x$ in the absence of flow gives, $M_x = M_{xo}exp(-t/T_2)$. $M_x$ becomes neglible when $t \gg T_2$ in CW NMR case with no flow of spins. Then one can easily obtain the expressions for $M_{Xo}$ and $M_{Yo}$ in the Laboratory frame as follows:

$$M_{Xo}=\frac{-Sin(wt)\gamma B_1 T_2 M_o}{1+\gamma^2 B_1^{\ 2}T_1 T_2} \qquad (40)$$

$$M_{Yo}=\frac{Cos(wt)\gamma B_1 T_2 M_o}{1+\gamma^2 B_1^{\ 2}T_1 T_2} \qquad (41)$$

Fortunately Eqns. (40) and (41) are the exact solutions of Bloch CW NMR magnetizations at resonance condition in the laboratory frames with no flow of spins (i.e., for stationary nuclear spins). This shows the correctness of our approach in formulating a coherent differntial Eqn. (37) that can describe the $M_y$ component of magnetization of flowing spins in the rotating frame under any form of rf $B_1$ field excitation. We believe that the Eqn. (37) can be applicable to all cases of NMR/MRI excitations of flowing spins whether pulsed rf or CW NMR. We believe that such formulations (Eqns. (36) or (37)) is not seen in literature before.

If an ideal imaginary situation can be assumed when $B_1(x,t)$ is both independent of time and x then the above equation (37 ) reduces to

$$\left( v\frac{\partial}{\partial x} + \frac{\partial}{\partial t} + \frac{1}{T_1} \right)v\frac{\partial M_y}{\partial x} + \left( v\frac{\partial}{\partial x} + \frac{\partial}{\partial t} + \frac{1}{T_1} \right)\frac{\partial M_y}{\partial t}$$

$$\tag{42}$$

$$+ \left( v\frac{\partial}{\partial x} + \frac{\partial}{\partial t} + \frac{1}{T_1} \right)\frac{M_y}{T_2} = \gamma\left( -\gamma M_y B_1 + \frac{M_o}{T_1} \right)B_1$$

Assuming flowing spin velocity $v$ to be both independent of time and x, and assuming

$$\frac{\partial}{\partial x}\frac{\partial M_y}{\partial t} = \frac{\partial}{\partial t}\frac{\partial M_y}{\partial x}$$

one gets from Eqn. (42)

$$v^2\frac{\partial^2 M_y}{\partial x^2} + v\frac{\partial^2 M_y}{\partial x\partial t} + \frac{v}{T_1}\frac{\partial M_y}{\partial x} + v\frac{\partial^2 M_y}{\partial x\partial t}$$

$$+ \frac{\partial^2 M_y}{\partial t^2} + \frac{1}{T_1}\frac{\partial M_y}{\partial t} + \frac{v}{T_2}\frac{\partial M_y}{\partial x} + \frac{1}{T_2}\frac{\partial M_y}{\partial t}$$

$$+ \frac{1}{T_1 T_2}M_y = -\gamma^2 B_1^2 M_y + \frac{\gamma B_1 M_o}{T_1}$$

or,

$$v^2\frac{\partial M_y}{\partial x^2} + 2v\frac{\partial^2 M_y}{\partial x\partial t} + v\left( \frac{1}{T_1} + \frac{1}{T_2} \right)\frac{\partial M_y}{\partial x}$$

$$+ \left( \frac{1}{T_1} + \frac{1}{T_2} \right)\frac{\partial M_y}{\partial t} + \frac{\partial^2 M_y}{\partial t^2} \tag{43}$$

$$+ \left( \frac{1}{T_1 T_2} + \gamma^2 B_1^2 \right)M_y = \frac{\gamma B_1 M_o}{T_1}$$

Eqn. (43) describing flow magnetization $M_y$ in the rotating frame is obtained from Eqns. (36)-(37) when $B_1$ field is independent of both t and x. At any given time t, we could get information from it about the system, provided that appropriate boundary conditions are applied. If in a NMR experiment the time dependence $\frac{\partial M_y}{\partial t}$ of $M_y$ can be dropped in Eqn. (39) and one gets the flow magnetization equation:

$$v^2\frac{\partial M_y}{\partial x^2} + v\left( \frac{1}{T_1} + \frac{1}{T_2} \right)\frac{\partial M_y}{\partial x} + \left( \frac{1}{T_1 T_2} + \gamma^2 B_1^2 \right)M_y = \frac{\gamma B_1 M_o}{T_1}$$

$$\tag{44}$$

as derived by Awojyogbe [10-12] earlier as the original flow magnetization equation [22]. They had assumed $B_1$ *to* be function of both x and t. We see from our above derivations that in Eqn. (44) the $B_1$ field must be independent of both x and t. Eqn. (44) has recently been discussed theoretically in several publications. We have shown above that Eqn. (44) can be derived from the mother Eqns. (36)-(37) of NMR flow magnetization based on conditions described above. Experimental situations conforming to such conditions of Eqn. (44) could be a situation in which the $B_1$ field, a CW NMR, is nearly uniform over the area of flow interest.

Thus we see that our Eqn. (36) or Eqn. (37) becomes the mother equation for NMR/MRI flow magnetization from which all equations of flow magnetization (in the rotating frame at resonance, i.e., when $\omega = \omega_o$) can be obtained depending on experimental situations i.e., rf pulse sequences, slice selection gradient pulses etc. In a given NMR/MRI experiment a particular slice is selected through the application of magnetic field gradient and rf pulse is applied and signal is recorded. Based on form of the rf pulse $B_1(x,t)$ field the flow magnetization and hence the blood flow signal across the slice or along a selected vessel can be computed from Eqns. (37), (38). This would enable us to obtain non-invasive flow information (appropriate for the medical diagnosis) accurately using the signal data. The approach given above could be extended to obtaining correct NMR/MRI spin flow magnetization equations in the Laboratory frame of reference. Such equations will be reported in our next work.

### 5.1.    For blood flow velocity along laboratory X direction

In the above formulations (Eqns. (36) and (37)) it was assumed that the flow velocity is along x direction, the direction of the rotating rf field. In a realistic MRI sequence of blood flow estimation the artery (through which flow is occuring) could be selected through the slice encoding pulsed gradient fields and the rf $B_1$ field could be aligned along or perpendicular to the flow direction (through the selected vessel). Let us assume that in such situation the flow velocity $V_o$ as well as the rf $B_1$ field is along the laboratory X direction. The

components of the flow velocity along the rotationg x and y directions are

$$v_x = V_o \cos\omega t \text{ and } v_y = V_o \sin\omega t \qquad (44a)$$

Then it can be shown easily by following the above methods the Bloch NMR/MRI flow magnetization equation (in the rotating frame) takes the form with magnetizing field $B_o$ being along Z direction

$$\left[\left(v \bullet \nabla + \frac{\partial}{\partial t}\right)\frac{1}{\gamma B_1(x,t)}\left(v \bullet \nabla + \frac{\partial}{\partial t} + \frac{1}{T_2}\right)\right.$$

$$\left. + \frac{1}{\gamma B_1(x,t)T_1}\left(v \bullet \nabla + \frac{\partial}{\partial t} + \frac{1}{T_2} + \gamma^2 B_1^2(x,t)T_1\right)\right]M_y$$

$$= \frac{M_o}{T_1} \qquad (45)$$

Where

$$v \bullet \nabla = v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y}$$

Eqn. (45) describes the flow magnetization in the rotating frame of reference with $v_x$ and $v_y$ obtained through the velocity $V_o$ in the Laboratory frame (see Eqn. (44a)). One can easily see from Eqn. (45) that for no flow conditions one obtains Eqn. (39) and consequently Eqns. (40) and (41) for CW NMR on static tissue or samples. Also Eqns. (36) and (37) emerge from (45) when $v_y$ is dropped (i.e., only $v_x$ is retained) as a special case. Solutions of Bloch NMR/MRI flow magnetizations in the rotating frame as given by Eqns. (36), (37) and (45) are beyond the scope of the present article and would be discussed subsequently. It is to be noted that the solutions depend on specific MRI sequences used in the blood flow estimation. The solutions of Eqns. (36), (37) and (45) may be attempted in line with the solutions proferred by Odoh and De [23]

## 5.2.    Slice selection in MRI

Questions may arise how to include magnetic field gradient used normally for tissue/vessel site selection? With the slice selection field gradients the resonance frequencies vary with tissue coordinates (x,y,z) depending on the bias magnetic field at the time of resonance. So, if the rf $B_1$ field is known at the point of consideration, then from the signal obtained one should be able to use the above equations to obtain the flow velocity (assumed to be along x direction) at that point.

In all the derivations above it is assumed that $B_1(x,t)$ is uniform in the yz plane This may not be the case in actual MRI imaging sequences or blood flow mapping. However, we believe that Eqns. (36), (37) or (45) can be applied in such cases also for computation of the magnetization over and hence the MRI signal from a given pixel, if the distributions of $B_1(x,t)$ over the yz plane is known.

## 5.3.    Applications of the above method of deriving Eqns. (36), (37) and (45) to diffusion MRI

Diffusion NMR/MRI equations are gaining importance recently [24]. Diffusion MRI was introduced in the mid-1980s [25]. Warach et al. [26] showed that the most successful application of diffusion MRI since the early 1990s has been in brain ischemia, following the discovery in cat brain by Moseley et al. [27] that water diffusion drops at a very early stage of the ischemic event. This pioneering work on diffusion anisotropy really took off with the introduction of the more rigorous formalism of the diffusion tensor by Basser et al. [28-30]. With diffusion tensor imaging (DTI), diffusion anisotropy effects in diffusion MRI data could be fully extracted, characterized, and exploited and could be providing even more exquisite details of tissue microstructure. Many studies have been published dealing with the optimization of the MRI sequences necessary to gain access to the diffusion tensor, the processing and display of DTI data, and, of course, potential applications. The most advanced application is certainly that of fibre tracking in the brain, which, in combination with functional MRI, might open a window onto the important issue of connectivity. Basser and Jones [31] showed that when implanted cells proliferate and become organized into tissue, the overall diffusion of water decreases because the molecules encounter additional physical barriers. Diffusion weighted MRI provides a means to identify the emerging tissue structure through the reduction of the apparent diffusion coefficient. However, so far diffusion MRI/NMR or diffusion magnetization with and without flow has not been given thorough theoretical foundation, especially when relaxation times are to be considered. We expect that our above methodology will provide the correct theoretical equation describing diffusion magnetization with flow and relaxation effects. It may be mentioned that Bloch Torrey equation [32] describes diffusion magnetization without

relaxation effects and flow, and therefore from the physical point of view it is not applicable to a realistic situation that can be studied by MRI. The reason is that whether magnetization undergoes diffusion with or without flow velocity it must be governed by relaxation effects (when in NMR/MRI both rf $B_1$ field and bias $B_o$ field are present). We shall report later the equations for $M_y$ and $M_x$ magnetizations when both diffusion and flow are present along with relaxation effects. Such formulations (not reported yet) then can open door for correct assessment of relevant diffusion parameters from the measured signals for application to medical diagnostics and petroleum industry.

## 6. Conclusion

We have derived NMR/MRI blood flow magnetization equation in the rotating frame of reference. This equation, which is the first of its kind, is expected to serve as the mother equation for the nuclear spin flow magnetization for all NMR/MRI experiments applicable to non-invasive blood flow estimation. We have shown that the corresponding equation, which is in literature and which is mostly due to the work of Awojyogbe and his group, can be derived from our original equation under correct assumptions. The method of deriving the equation can be extended to NMR/MRI spin flow magnetization equation in the Laboratory frame and to Bloch-Torey equation for inclusion of flow of spins and relaxation times. It is needless to mention that the correct signal expression can be derived and the flow signal computed, once the correct dependence of magnetization components of flowing nuclear spins on flow velocity, rf $B_1$ field, spatial coordinates, relaxation times and other entities is known. This is expected to enhance the accuracy of blood flow estimation by NMR/MRI experiments.

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