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Scattering Matrix Formalism for Spin s (s=0, 1/2) Particles

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In this work, we consider the problem of a relativistic spin-s (0, 1/2) particles interacting with a one-dimensional symmetrical scalar potential using the spin-s Klein Gordon equation. In the scattering case, we construct the formalism of scattering matrix for the spin-s Klein Gordon relativistic particle in a symmetric potential. Through the scattering matrix, we can derive the phase shift, the scattering amplitude and consequently reflection and transmission coefficients. Finally, we applied the results obtained to a symmetric scalar potential; this potential is that of cusp.

1. Introduction

The study of scattering plays a very important role in modern physics because it offers valuable insights into the nature of the interactions between particles coming into contact. This theory finds its origins in theories of classical and quantum mechanics. In classical physics, the state of the incoming (free) particle is entirely determined by its momentum, which is equally true for the outgoing particle. At the quantum level, it is not generally possible to predict with certainty which end state will result from a given collision. We are therefore only trying to predict the probabilities for a certain final state. The problem is to establish the relation between the initial state Ψ^{in} and the final state Ψ^{out}. In quantum mechanics, the knowledge scattering operator $\hat{\mathbf{S}}$ allows us to determine the final state Ψ^{out} from any initial state Ψ^{in}. The corresponding mathematical object is the scattering matrix $\hat{S}$, the S-matrix can be defined as the matrix that transforms the coefficients of the incoming waves into those of the outgoing waves $\Psi^{out} = \hat{\mathbf{S}}\Psi^{in}$.

The purpose of this article is to address some scattering problems of relativistic particles of spin-s $(s = 0, 1/2)$ interacting with a symmetric scalar potential. The one-dimensional scattering problem has been studied in terms of the S-matrix by a number of authors (see, for example, [1-6]).

The manuscript has four sections. The second section introduces the spin-s Klein Gordon formalism for a charged particle coupled to an electromagnetic field. The spin-s Klein Gordon equation combines Klein Gordon and Dirac equations; Klein Gordon equation is used to describe spin zero particles [7] whereas the Dirac equation for the spin 1/2 particles in relativistic quantum mechanics [8, 9]. In the third section, we express the scattering properties in terms of the S-matrix and determine the element of scattering matrix by using $C_sP_sT_s$ symmetry and the charge conservation. From scattering matrix, we drew phase shifts, the scattering amplitude, and reflection and transmission coefficients for a symmetrical scalar potential. In the fourth section, the results and discussion are presented. Finally, in the fifth section, we present analytical solution of the spin-s Klein-like many studies [10-15], the exact solutions to spin-s, the Gordon equation with a Cusp potential in one dimension, where the Klein Gordon equation are given in terms of Whittaker's functions. In the scattering case, we obtain the scattering matrix, phase shifts the scattering amplitude and hence the reflection and transmission coefficients for the Cusp potential.

2. The Klein Gordon formalism for spin-s particles

In relativistic quantum mechanics, the Klein Gordon and the Dirac equation describe the massive particles of spin 0 and $1/2$, respectively. In presence of an electromagnetic field, $A_\mu\left(A_0, \vec{A}\right)$ the Klein Gordon equation is given by ($\hbar = c = 1$)

$$\left\{\left(\frac{\partial}{\partial t} + ieA_0\right)^2 - \left(\vec{\nabla} - ie\vec{A}\right)^2 + m^2\right\}\Psi_0\left(\vec{r}, t\right) = 0, \tag{1}$$

The Dirac equation is given by

$$\left(\not{p} - e\not{A} - m\right)\Psi_{1/2}\left(\vec{r}, t\right) = 0, \tag{2}$$

Where, $\vec{r} = (x, y, z)$, A_0 and $\vec{A}$ are the four-vector A_μ. $\not{p} - e\not{A} = \gamma^\mu\left(i\partial_\mu - eA_\mu\right)$, and γ^μ ($\mu = 0,1,2,3$) are gamma matrices.

By making the substitution

$$\Psi_s\left(\vec{r}, t\right) = \left(\not{p} - e\not{A} + m\right)^{2s}\Phi_s\left(\vec{r}, t\right), \quad s = 0, \frac{1}{2} \tag{3}$$

If we adopt the following notation for any matrix A

$$\left(A\right)^{2s} = \begin{cases} 1 & \text{for} \quad s = 0 \\ A & \text{for} \quad s = 1/2, \end{cases} \tag{4}$$

In this case, Eqns. (1) and (2) can be expressed by a single relation, called the spin-s Klein Gordon (KG - s) equation

$$\left\{\left[\left(\frac{\partial}{\partial t}+ieA_0\right)^2-\left(\vec{\nabla}-ie\vec{A}\right)^2+m^2\right]I_{2^{2s}(2s+1)}+se(\sigma F)^{2s}\right\}\Phi_s(\vec{r},t)=0, \quad (5)$$

Here, $\Phi_s(\vec{r},t)$ is a wave function with $2^{2s}(2s+1)$ components and $I_{2^{2s}(2s+1)}$ is the identity matrix of dimension $2^{2s}(2s+1)$ and F is the electromagnetic tensor defined by the components $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ and $\sigma F=\sigma^{\mu\nu}F_{\mu\nu}$ with $\sigma^{\mu\nu}=\frac{i}{2}\left[\gamma^\mu,\gamma^\nu\right]$.

For $s=0$, Eqn. (5) reduces to the Klein-Gordon equation and for $s=1/2$; we obtain the quadratic form of Dirac equation (or Klein-Gordon spin-1/2).
Moreover, starting from Eqn. (5), we define a continuity equation:

$$\frac{\partial J_s^0}{\partial t}+\vec{\nabla}\vec{J}_s=0, \quad (6)$$

Where, J_s^0 and $\vec{J}_s$ are given by:

$$J_s^0=\frac{1}{2im}\left[\overline{\Phi}_s(\vec{r},t)\left(\frac{\partial\Phi_s(\vec{r},t)}{\partial t}\right)-\left(\frac{\partial\overline{\Phi}_s(\vec{r},t)}{\partial t}\right)\Phi_s\right]+\frac{e}{m}A^0\overline{\Phi}_s(\vec{r},t)\Phi_s(\vec{r},t), \quad (7)$$

And

$$\vec{J}_s=\frac{1}{2im}\left[\overline{\Phi}_s(\vec{r},t)\left(\vec{\nabla}\Phi_s(\vec{r},t)\right)-\left(\vec{\nabla}\overline{\Phi}_s(\vec{r},t)\right)\Phi_s(\vec{r},t)\right]+\frac{e}{m}A\overline{\Phi}_s(\vec{r},t)\Phi_s(\vec{r},t), \quad (8)$$

Here, $\overline{\Phi}_s(\vec{r},t)=\left(\Phi_s(\vec{r},t)\right)^\dagger\left(\gamma^0\right)^{2s}$ denotes the adjoint and $\left(\Phi_s(\vec{r},t)\right)^\dagger=\left(\left(\Phi_s(\vec{r},t)\right)^*\right)^T$.
In addition, the scalar product is defined by

$$\langle\Phi_s\mid\Phi_s\rangle=\int\overline{\Phi}_s(\vec{r},t)\left[\left(\frac{\partial}{\partial t}+eA^0\right)\Phi_s(\vec{r},t)\right]-\left[\left(\frac{\partial}{\partial t}-eA^0\right)\overline{\Phi}_s(\vec{r},t)\right]\Phi_s(\vec{r},t)d^3r. \quad (9)$$

If we write $\Phi_s(\vec{r},t)=\left(\Phi_s^-(\vec{r},t),\Phi_s^+(\vec{r},t)\right)^T$, and choose for the γ^μ the Weyl representation [16]:

$$\gamma^0=\begin{pmatrix}0 & I_2\\ I_2 & 0\end{pmatrix},\quad \vec{\gamma}=\begin{pmatrix}0 & -\vec{\sigma}\\ \vec{\sigma} & 0\end{pmatrix}, \quad (10)$$

Where, $\Phi_s^-(\vec{r},t)$ and $\Phi_s^+(\vec{r},t)$ are a wave function with 2^{2s} components, and $\vec{\sigma}=(\sigma_1,\sigma_2,\sigma_3)$ are the usual Pauli matrices. With this representation, Eqn. (5) decomposes into a system of two coupled equations:

$$\begin{pmatrix}H_s^-(\vec{r},t) & 0\\ 0 & H_s^+(\vec{r},t)\end{pmatrix}\begin{pmatrix}\Phi_s^-(\vec{r},t)\\ \Phi_s^+(\vec{r},t)\end{pmatrix}=0, \quad (11)$$

Here,

$$H_s^\mp(\vec{r},t)=\left[\left(\frac{\partial}{\partial t}+ieA_0\right)^2-\left(\vec{\nabla}-ie\vec{A}\right)^2+m^2\right]I_{2^{2s}}\mp 2ise(\vec{\sigma})^{2s}\left(\vec{E}\pm i\vec{B}\right) \quad (12)$$

$\vec{E}$ and $\vec{B}$ are the electric and magnetic fields. After giving the elements constituting the Klein Gordon formalism for spin-s particles, let us now turn to the construction of the one-dimensional scattering matrix $\hat{S}_{KG-s}$.

3. Scattering matrix in a symmetrical potential

We consider a relativistic particle of mass m, energy E, spin- s and charge e in a one-dimensional model moving under the effect of a symmetrical scalar potential $A_0(x,t)=V(x)$ and with infinite behavior $V(x)\overset{|x|\to\infty}{\to}0$. In this case, $\vec{E}$ and $\vec{B}$ become:

$$\vec{E}=-\frac{dV(x)}{dx}\vec{\imath},\ \vec{B}=\vec{0}, \quad (13)$$

So, the system of equations (Eqn. (11)) is written as follows

$$\begin{pmatrix}H_s^-(x,t) & 0\\ 0 & H_s^+(x,t)\end{pmatrix}\begin{pmatrix}\Phi_s^-(x,t)\\ \Phi_s^+(x,t)\end{pmatrix}=0, \quad (14)$$

Here,

$$H_s^\mp(x,t)=\left[\left(i\frac{\partial}{\partial t}-eV(x)\right)^2+\frac{d^2}{dx^2}-m^2\right]I_{2^{2s}}\pm 2ise(\sigma_1)^{2s}\frac{dV(x)}{dx}. \quad (15)$$

As the potential $V(x)$ is time independent, we can write the solutions of Eqn. (14) under the form $\Phi_s^\mp(x,t)=e^{-iEt}\Phi_s^\mp(x)$, consequently, $\Phi_s^\mp(x)$ satisfies

$$\left\{D_{KG}^2 I_{2^{2s}}\pm 2ies(\sigma_1)^{2s}\frac{dV(x)}{dx}\right\}\Phi_s^\mp(x)=0. \quad (16)$$

$D_{KG}^2=\frac{d^2}{dx^2}+(E-eV(x))^2-m^2$ is the operator relative to KG-0.

At great distance from the diffuser $V(x)\overset{|x|\to\infty}{\to}0$, the asymptotic form of the wave functions $\Phi_{s,L}^\mp(x)$ (arrival of $(-\infty)$) and $\Phi_{s,R}^\mp(x)$ (arrival of $+\infty$), corresponding to scattering states $\left(E^2-m^2>0\right)$, can be written [5, 6, 17] as

$$\Phi_{s,L}^\mp(x)=\begin{cases}\mathbf{1}_s e^{ikx}+R_{s,L}e^{-ikx} & x\to-\infty\\ T_{s,L}e^{ikx} & x\to+\infty\end{cases}, \quad (17)$$

$$\Phi^{\mp}_{s,R}(x)=\begin{cases}\mathbf{1}_s e^{-ikx}+R_{s,R}e^{+ikx} & x\to+\infty\\ T_{s,R}e^{-ikx} & x\to-\infty\end{cases}. \qquad (18)$$

Where, $\mathbf{1}_s=(1_s^-,1_s^+)^T$ is unit vector of $2^{2s}(2s+1)$ components $T_{s,L}$, $T_{s,R}$, $R_{s,L}$ and $R_{s,R}$ are vectors of $2^{2s}(2s+1)$ components.

Now, examine the general problem. The stationary wave function of Eqn. (16) [5, 6] and [17] is written as

$$\Phi^{\mp}_{s}(x)=\begin{cases}\Phi^{\mp}_s(out,+\infty)e^{ikx}+\Phi^{\mp}_s(in,+\infty)e^{-ikx} & x\to+\infty\\ \Phi^{\mp}_s(in,-\infty)e^{ikx}+\Phi^{\mp}_s(out,-\infty)e^{-ikx} & x\to-\infty\end{cases} \qquad (19)$$

Where, $\Phi^{\mp}_s(in,out,\pm\infty)$ are vectors with 2^{2s} components, and $k=\sqrt{E^2-m^2}$.

Let us consider that the incoming and outgoing parts of the wave function $\Phi^{\mp}_s(x)$ are given by:

$$\begin{cases}\Phi^{\mp,in}_s(x)=\Phi^{\mp}_s(in,-\infty)e^{ikx}\theta(-x)+\Phi^{\mp}_s(in,+\infty)e^{-ikx}\theta(x)\\ \Phi^{\mp,out}_s(x)=\Phi^{\mp}_s(out,-\infty)e^{-ikx}\theta(-x)+\Phi^{\mp}_s(out,+\infty)e^{ikx}\theta(x)\end{cases}$$

(20)

By definition the matrix $\hat{S}_{KG-s}$ connects $\Phi^{out}_s(x)$ to $\Phi^{in}_s(x)$ [5, 6] by

$$\begin{pmatrix}\Phi^-_s(out,+\infty)\\ \Phi^+_s(out,+\infty)\\ \Phi^-_s(out,-\infty)\\ \Phi^+_s(out,-\infty)\end{pmatrix}=\begin{pmatrix}\hat{s}_{11} & \hat{s}_{12} & \hat{s}_{13} & \hat{s}_{14}\\ \hat{s}_{21} & \hat{s}_{22} & \hat{s}_{23} & \hat{s}_{24}\\ \hat{s}_{31} & \hat{s}_{32} & \hat{s}_{33} & \hat{s}_{34}\\ \hat{s}_{41} & \hat{s}_{42} & \hat{s}_{43} & \hat{s}_{44}\end{pmatrix}\begin{pmatrix}\Phi^-_s(in,-\infty)\\ \Phi^+_s(in,-\infty)\\ \Phi^-_s(in,+\infty)\\ \Phi^+_s(in,+\infty)\end{pmatrix} \qquad (21)$$

Here $\hat{s}_{ij}$ are matrices of dimension 2^{2s}. In the case of spin 0, the system reduce to

$$\begin{pmatrix}\Phi_0(out,+\infty)\\ \Phi_0(out,-\infty)\end{pmatrix}=\begin{pmatrix}\hat{s}_{11} & \hat{s}_{13}\\ \hat{s}_{31} & \hat{s}_{33}\end{pmatrix}\begin{pmatrix}\Phi_0(in,-\infty)\\ \Phi_0(in,+\infty)\end{pmatrix}, \qquad (22)$$

Where, $\Phi_0(in,out,\pm\infty)$ are coefficients.

Eqns. (21) and (22) can be unified into the following formula

$$\Phi_s(out)=\hat{S}_{KG-s}\Phi_s(in), \qquad (23)$$

Where, the vectors $\Phi_s(in,out)$ and the $\hat{S}_{KG-s}$ the matrix has respectively the dimension $2^{2s+1}(2s+1)$.

In fact, matrices $\hat{s}_{ij}$ are not all independent but verify some constraints derived from general conditions reflecting the symmetry of the KG-s equation. Consider $C_sP_sT_s$ symmetry is the product of three fundamental operators: parity P_s, time reversal T_s and charge conjugation C_s, these operators are characterized by the following properties:

$$C_sC_s^{-1}=P_sP_s^{-1}=T_sT_s^{-1}=I_{2^{2s}(2s+1)} \qquad (24)$$

Any operator is determined at a phase factor closure, we will take afterwards $\eta_C=\eta_P=\eta_T=i$, so that $|\eta_C|^2=|\eta_P|^2=|\eta_T|^2=1$

Firstly, we apply the operator P_s, which transforms $(x,t)\to(-x,t)$; by changing $x\to-x$ in Eqn. (14), we get:

$$\begin{pmatrix}H^-_s(-x,t) & 0\\ 0 & H^+_s(-x,t)\end{pmatrix}\begin{pmatrix}\Phi^-_s(-x,t)\\ \Phi^+_s(-x,t)\end{pmatrix}=0. \qquad (25)$$

Multiplying Eqn. (25) on the left by P_s, and using Eqn. (24), we find

$$P_s\begin{pmatrix}H^-_s(-x,t) & 0\\ 0 & H^+_s(-x,t)\end{pmatrix}P_s^{-1}P_s\begin{pmatrix}\Phi^-_s(-x,t)\\ \Phi^+_s(-x,t)\end{pmatrix}=0. \qquad (26)$$

Posing

$$\Phi^{P_s}_s(x,t)=P_s\Phi_s(-x,t)=P_s\begin{pmatrix}\Phi^-_s(-x,t)\\ \Phi^+_s(-x,t)\end{pmatrix}, \qquad (27)$$

$\Phi^{P_s}_s(x,t)$ is a solution of Eqn. (14) if P_s satisfies the condition

$$P_s\begin{pmatrix}H^-_s(-x,t) & 0\\ 0 & H^+_s(-x,t)\end{pmatrix}P_s^{-1}=\begin{pmatrix}H^-_s(x,t) & 0\\ 0 & H^+_s(x,t)\end{pmatrix} \qquad (28)$$

This condition is true if $P_s=i(\gamma^0)^{2s}$.

Using Eqn. (19) and Eqn. (27) we move $\Phi_s(x,t)$ to $\Phi^P_s(x,t)$ and making the following changes:

$$\begin{aligned}\Phi^{\mp}_s(out,\pm\infty)&\leftrightarrow i\Phi^{\pm}_s(out,\mp\infty),\\ \Phi^{\mp}_s(in,\pm\infty)&\leftrightarrow i\Phi^{\pm}_s(in,\mp\infty).\end{aligned} \qquad (29)$$

and transporting this last expression in (21), we find

$$\begin{aligned}&\hat{s}_{11}=\hat{s}_{44},\ \hat{s}_{33}=\hat{s}_{22},\ \hat{s}_{12}=\hat{s}_{43},\ \hat{s}_{31}=\hat{s}_{24},\\ &\hat{s}_{13}=\hat{s}_{42},\ \hat{s}_{23}=\hat{s}_{32},\ \hat{s}_{21}=\hat{s}_{34},\ \hat{s}_{41}=\hat{s}_{14}.\end{aligned} \qquad (30)$$

Let us perform again operation T_s, which changes $(x,t)\to(x,-t)$; by changing $t\to-t$, and taking the conjugate complex of Eqn. (14), we find

$$\begin{pmatrix}(H^-_s(x,-t))^* & 0\\ 0 & (H^+_s(x,-t))^*\end{pmatrix}\begin{pmatrix}(\Phi^-_s(x,-t))^*\\ (\Phi^+_s(x,-t))^*\end{pmatrix}=0. \qquad (31)$$

Multiplying (31) on the left by T_s, and using (24) we obtain

$$T_s\begin{pmatrix}(H^-_s(x,-t))^* & 0\\ 0 & (H^+_s(x,-t))^*\end{pmatrix}T_s^{-1}T_s\begin{pmatrix}(\Phi^-_s(x,-t))^*\\ (\Phi^+_s(x,-t))^*\end{pmatrix}=0, \qquad (32)$$

Putting

$$\Phi^{T_s}_s(x,t)=T_s\Phi^*_s(x,-t)=T_s\begin{pmatrix}(\Phi^-_s(x,-t))^*\\ (\Phi^+_s(x,-t))^*\end{pmatrix} \qquad (33)$$

$\Phi^{T_s}_s(x,t)$ satisfies Eqn. (14), if T_s satisfies the condition:

$$T_s\begin{pmatrix}\left(H_s^-(x,-t)\right)^* & 0\\ 0 & \left(H_s^+(x,-t)\right)^*\end{pmatrix}T_s^{-1}=\begin{pmatrix}H_s^-(x,t) & 0\\ 0 & H_s^+(x,t)\end{pmatrix}. \quad (34)$$

Eq. (34) is satisfied by the operator $T_s=i\left(\gamma^1\gamma^3\right)^{2s}$.

From (19) and (33), we can go from $\Phi_s(x,t)$ to $\Phi_s^T(x,t)$ by carrying out the following change:

$$\begin{aligned}\Phi_s^{\mp}(out,\pm\infty)&\leftrightarrow-(\sigma_2)^{2s}\left(\Phi_s^{\mp}(in,\pm\infty)\right)^*,\\ \Phi_s^{\mp}(in,\pm\infty)&\leftrightarrow-(\sigma_2)^{2s}\left(\Phi_s^{\mp}(out,\pm\infty)\right)^*.\end{aligned} \quad (35)$$

Finally, let us apply the operator C_s , which transforms $e\to-e$; by changing $e\to-e$, and taking the Hermitian conjugate of Eqn. (14), we find

$$\begin{pmatrix}\Phi_s^-(x,t)\\ \Phi_s^+(x,t)\end{pmatrix}^\dagger\begin{pmatrix}H_s^-(x,t) & 0\\ 0 & H_s^+(x,t)\end{pmatrix}=0. \quad (36)$$

Multiplying (36) by $\left(\gamma^0\right)^{2s}$ from the right and using $\left(\left(\gamma^0\right)^{2s}\right)^2=I_{2^{2s}(2s+1)}$, we get

$$\overline{\Phi}_s(x,t)\begin{pmatrix}H_s^+(x,t) & 0\\ 0 & H_s^-(x,t)\end{pmatrix}=0, \quad (37)$$

Transposing Eqn. (37) and multiplying it from the left by C_s, we obtain

$$C_s\begin{pmatrix}H_s^+(x,t) & 0\\ 0 & H_s^-(x,t)\end{pmatrix}C_s^{-1}C_s\left(\begin{pmatrix}\Phi_s^-(x,t)\\ \Phi_s^+(x,t)\end{pmatrix}\left(\gamma^0\right)^{2s}\right)^\dagger=0, \quad (38)$$

Suppose

$$\Phi_s^C(x,t)=C_s\overline{\Phi}_s^T(x,t)=C_s\left(\begin{pmatrix}\Phi_s^-(x,t)\\ \Phi_s^+(x,t)\end{pmatrix}\left(\gamma^0\right)^{2s}\right)^\dagger, \quad (39)$$

for $\Phi_s^C(x,t)$ to be a solution to Eqn. (14) , C_s must satisfy the conditions:

$$C_s\begin{pmatrix}H_s^+(x,t) & 0\\ 0 & H_s^-(x,t)\end{pmatrix}C_s^{-1}=\begin{pmatrix}H_s^-(x,t) & 0\\ 0 & H_s^+(x,t)\end{pmatrix}, \quad (40)$$

Eqn. (40) is verified by $C_s=i\left(\gamma^2\gamma^0\right)^{2s}$.

Using Eqns. (19) and (39), we move from $\Phi_s(x,t)$ to $\Phi_s^C(x,t)$ through the following changes

$$\begin{aligned}\Phi_s^{\mp}(out,\pm\infty)&\leftrightarrow\mp i(\sigma_2)^{2s}\left(\Phi_s^{\pm}(in,\pm\infty)\right)^*,\\ \Phi_s^{\mp}(in,\pm\infty)&\leftrightarrow\mp i(\sigma_2)^{2s}\left(\Phi_s^{\pm}(out,\pm\infty)\right)^*.\end{aligned} \quad (41)$$

Combining Eqns. (29) , (35) , (41) and (21) we obtain

$$s_{12}=s_{21}=s_{14}=s_{41}=s_{23}=s_{32}=s_{34}=s_{43}=\mathbf{0}. \quad (42)$$

In addition, charge conservation gives:

$$\langle\Phi_s^{out}\mid\Phi_s^{out}\rangle=\langle\Phi_s^{in}\mid\Phi_s^{in}\rangle. \quad (43)$$

The last equation leads to the next relationship

$$\left(I_{2^{2s}}\otimes\left(\gamma^0\right)^{2s}\right)\hat{S}_{KG-s}^\dagger\left(I_{2^{2s}}\otimes\left(\gamma^0\right)^{2s}\right)\hat{S}_{KG-s}=I_{2^{(2s+1)}(2s+1)}, \quad (44)$$

with $\hat{S}_{KG-s}^\dagger=\left(\hat{S}_{KG-s}^*\right)^T$, and $\otimes$ being the tensor product.

By combining the relations deduced from, the invariances with respect to the operations P_s,. T_s , C_s and the charge conservation, we obtain the final form of the scattering matrix of a spin-s $(s=0,1/2)$ relativistic particle in a symmetrical potential:

$$\hat{S}_{KG-s}=\begin{pmatrix}T_s^+ & 0 & R_s^+ & 0\\ 0 & T_s^- & 0 & R_s^-\\ R_s^- & 0 & T_s^- & 0\\ 0 & R_s^+ & 0 & T_s^+\end{pmatrix} \quad (45)$$

$T_s^{\mp}$ and $R_s^{\mp}$ are matrices to $\left(2^{2s}\times2^{2s}\right)$ are given by:

$$\begin{aligned}T_s^+&=\begin{pmatrix}t_s^- & t_s^+\\ t_s^+ & t_s^-\end{pmatrix},\ T_s^-=\begin{pmatrix}t_s^- & -t_s^+\\ -t_s^+ & t_s^-\end{pmatrix},\\ R_s^+&=\begin{pmatrix}r_s^- & r_s^+\\ r_s^+ & r_s^-\end{pmatrix},\ R_s^-=\begin{pmatrix}r_s^- & -r_s^+\\ -r_s^+ & r_s^-\end{pmatrix},\end{aligned} \quad (46)$$

$t_s^{\pm}$ and $r_s^{\pm}$ are functions generally dependent on the potential shape.

For $s=0$ noting that $t_s^+=r_s^+=0$, so $T_0^+=T_0^-$ and $R_0^+=R_0^-$, in this case $\hat{S}_{KG-0}$ is symmetrical. For $s=1/2$ we get $(\sigma_3)\left(T_{1/2}^{\pm}\right)(\sigma_3)=T_{1/2}^{\mp}$ and $(\sigma_3)\left(R_{1/2}^{\pm}\right)(\sigma_3)=R_{1/2}^{\mp}$, so $\hat{S}_{KG-1/2}$ is pseudo-symmetrical. Theses latte r properties are specific to the spinning particle.

Let us now express the matrix $\hat{S}_{KG-s}$ in the partial wave basis, even and odd waves. To illustrate this, we apply the unitary transformation [5, 6]

$$\begin{pmatrix}\chi_s^-(in,-\infty)\\ \chi_s^+(in,-\infty)\\ \chi_s^-(in,+\infty)\\ \chi_s^+(in,+\infty)\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}I_{2^{2s}} & 0 & I_{2^{2s}} & 0\\ 0 & I_{2^{2s}} & 0 & I_{2^{2s}}\\ iI_{2^{2s}} & 0 & -iI_{2^{2s}} & 0\\ 0 & iI_{2^{2s}} & 0 & -iI_{2^{2s}}\end{pmatrix}\begin{pmatrix}\Phi_s^-(in,-\infty)\\ \Phi_s^+(in,-\infty)\\ \Phi_s^-(in,+\infty)\\ \Phi_s^+(in,+\infty)\end{pmatrix} \quad (47)$$

And

$$\begin{pmatrix}\chi_s^-(out,-\infty)\\ \chi_s^+(out,-\infty)\\ \chi_s^-(out,+\infty)\\ \chi_s^+(out,+\infty)\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}I_{2^{2s}} & 0 & I_{2^{2s}} & 0\\ 0 & I_{2^{2s}} & 0 & I_{2^{2s}}\\ iI_{2^{2s}} & 0 & -iI_{2^{2s}} & 0\\ 0 & iI_{2^{2s}} & 0 & -iI_{2^{2s}}\end{pmatrix}\begin{pmatrix}\Phi_s^-(out,-\infty)\\ \Phi_s^+(out,-\infty)\\ \Phi_s^-(out,+\infty)\\ \Phi_s^+(out,+\infty)\end{pmatrix}, \quad (48)$$

Where, $\chi_s^{\mp}(in,\pm\infty)$ and $\chi_s^{\mp}(out,\pm\infty)$ are vectors with 2^{2s} components. By combining Eqns. (47) , (48) and (21) it comes to:

$$\begin{pmatrix}\chi_s^-(out,+\infty)\\ \chi_s^+(out,+\infty)\\ \chi_s^-(out,-\infty)\\ \chi_s^+(out,-\infty)\end{pmatrix}=M_{KG-s}\begin{pmatrix}\chi_s^-(in,-\infty)\\ \chi_s^+(in,-\infty)\\ \chi_s^-(in,+\infty)\\ \chi_s^+(in,+\infty)\end{pmatrix} \quad (49)$$

Where, M_{KG-s} is the scattering matrix in the partial wave base. Using Eqn. (44) we obtain

$$M_{KG-s}=I_{2^{2s}}\otimes\begin{pmatrix}\exp\left(i\delta_s^0\right) & 0\\ 0 & \exp\left(i\delta_s^1\right)\end{pmatrix}, \tag{50}$$

Here $\delta_s^l,\ l=0,1$ represents respectively the phase shift of the event and odd waves, given by:

$$\exp\left(i\delta_s^l\right)=t_s^-+(-1)^l\sqrt{\left(t_s^+\right)^2+\left(r_s^-\right)^2-\left(r_s^+\right)^2}. \tag{51}$$

For the wave function, let's combine Eqns. (47) and (48) with Eqn. (21) and insert the result in Eqn. (19) to obtain the following:

$$\Phi_s^{\mp}(x)=\frac{1}{\sqrt{2}}\left[\chi_s^{\mp}(in,-\infty)\psi_s^0(x)+\chi_s^{\mp}(in,+\infty)\psi_s^1(x)\right], \tag{52}$$

With

$$\psi_s^l(x)=2(i\varepsilon)^l e^{i\delta_s^l}\left[\cos\left(k\,|\,x\,|+\delta_s^l+\frac{l\pi}{2}\right)\right],\quad l=0,1, \tag{53}$$

Where, $\varepsilon=+(-)$ according to $x>0$ $(x<0)$.

For a particle spreads $-\infty\rightarrow+\infty$, with (19), $\Phi_s^{\mp}(in,+\infty)=\mathbf{0}$, in these conditions $\chi_s^{\mp}(in,+\infty)=i\chi_s^{\mp}(in,-\infty)$. So, Eqn. (52) becomes

$$\Phi_{s,L}^{\mp}(x)=\begin{cases}\sqrt{2}\chi_s^{\mp}(in,-\infty)\left[e^{ikx}+f_s^- e^{-ikx}\right] & x\rightarrow-\infty\\ \sqrt{2}\chi_s^{\mp}(in,-\infty)\left[e^{ikx}+\left(1+f_s^+\right)e^{ikx}\right] & x\rightarrow+\infty\end{cases} \tag{54}$$

with $f_s^{\mp}$ are the scattering amplitude, given by:

$$f_s^{\mp}=f_s^0\mp f_s^1,\ f_s^l=\frac{1}{2}\left(e^{2i\delta_s^l}-1\right)=ie^{i\delta_s^l}\sin\delta_s^l,\ l=0,1. \tag{55}$$

To calculate the reflection and transmission coefficients we use Eqns. (54) and (55) to get

$$\begin{aligned}\mathrm{T}_s&=|1+f_s^+|^2=\frac{1}{4}\left|e^{2i\delta_s^0}+e^{2i\delta_s^1}\right|^2,\\ \mathrm{R}_s&=|f_s^-|^2=\frac{1}{4}\left|e^{2i\delta_s^0}-e^{2i\delta_s^1}\right|^2.\end{aligned} \tag{56}$$

4. Application

We will consider as application of the previous results the case of a relativistic particle of mass m, energy E, spin-s and charge e diffused by the Cusp's potential, defined by [10]:

$$V(x)=V_0 e^{-\frac{|x|}{a}}, \tag{57}$$

Where, V_0 and a are real positives. Here, V_0 represents the height of the potential and the parameter a defines the shape of the potential.

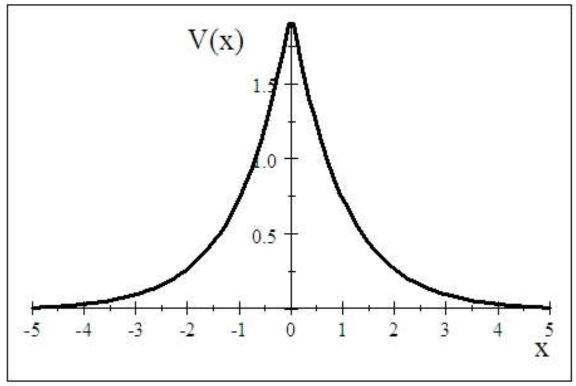

Figure 1: Cusp Potential for $a = 1$, $V_0 = 2$

In the limiting case, $a\rightarrow0$, this potential reduces to a repulsive delta interaction of strength $2aV_0$ [10]. On the other hand, this potential models an attractive center where $V(x)$ varies from $V(x)\rightarrow0$ for $|x|\rightarrow\infty$ to $V(x)\rightarrow V_0$ for $x\rightarrow0$. This is an atom model (Coulomb potential) [10].

The particle is subjected to a time-independence and in this case the solution of Eqn. (14) is written as $\Phi_s^{\mp,c}(x,t)=e^{-iEt}\Phi_s^{\mp,c}(x)$, consequently, $\Phi_s^{\mp,c}(x)$ satisfies

$$\left\{D_{KG}^{2,c}I_{2^{2s}}\pm 2ies(\sigma_1)^{2s}\frac{d}{dx}\left(V_0 e^{-\frac{|x|}{a}}\right)\right\}\Phi_s^{\mp,c}(x)=0. \tag{58}$$

Where, $D_{KG}^{2,c}=\frac{d^2}{dx^2}+\left(E-eV_0 e^{-\frac{|x|}{a}}\right)^2-m^2$. To decouple the system Eqn. (58) uses the following transformation

$$\Phi^{\mp,c}(x)=\left[\frac{\sigma_3+\sigma_1}{2}\right]^{2s}\chi_s^{\mp}(x),\ V(x)=V_0 e^{-\frac{|x|}{a}}, \tag{59}$$

We now obtain

$$\left[D_{KG}^{2,c}I_{2^{2s}}\pm 2ise(\sigma_3)^{2s}\frac{d}{dx}\left(V_0 e^{-\frac{|x|}{a}}\right)\right]\chi_s^{\mp}(x)=0. \tag{60}$$

We treat here the case $\left(E^2-m^2>0\right)$, corresponding to scattering states. Since there is an absolute value for variable x, let us distinguish as usual the two regions $x<0$ and $x>0$.

For $x<0$, by the change of variable $y=2iaV_0 e^{x/a}$, we transform the system (Eqn. (60) into

$$\left\{\left(y\frac{d}{dy}\left(y\frac{d}{dy}\right)-\left(iaE-\frac{y}{2}\right)^2+m^2a^2\right)I_{2^{2s}}\pm s(\sigma_3)^{2s}y\right\}\chi_s^{\mp}(x)=0. \tag{61}$$

We now introduce the change $\chi_s^{\mp}(x)=y^{-\frac{1}{2}}f_s^{\mp}(y)$ and we obtain for $f_s^{\mp}(y)$ a Whittaker equation [20, 21]

$$\left\{\left(\frac{d^2}{dy^2}-\frac{1}{4}+\frac{iaE}{y}+\frac{1/4-\left(ia\sqrt{E^2-m^2}\right)^2}{y^2}\right)I_{2^{2s}}\pm\frac{s(\sigma_3)^{2s}}{y}\right\}f_s^{\mp}(y)=0, \tag{62}$$

The solution of which is a combination of Whittaker's functions

$$f_s^{\mp}(y)=\begin{pmatrix} C_1^{\mp}M_{\kappa_s^{\pm},-\mu}(y)+C_2^{\mp}M_{\kappa_s^{\pm},\mu}(y) \\ D_1^{\mp}M_{\kappa_s^{\pm},-\mu}(y)+D_2^{\mp}M_{\kappa_s^{\pm},\mu}(y) \end{pmatrix} \quad (63)$$

Where, $M_{\kappa_s^{\pm},\pm\mu}(y)$, $\kappa_s^{\pm}$ and μ are given by:

$$M_{\kappa_s^{\pm},\pm\mu}(y)=y^{1/2\pm\mu}e^{-y/2}\ {}_1F_1(1/2-\kappa_s^{\pm}\pm\mu,1\pm 2\mu,y), \\ \kappa_s^{\pm}=iaE\pm s,\ \mu=iak. \quad (64)$$

and ${}_1F_1(1/2-\kappa_s^{\pm}\pm\mu,1\pm 2\mu,y)$ is the confluent hypergeometric function [20, 21].

As we proceed to the other region $x>0$, and by making the change $y=2iaV_0e^{-x/a}$ and replaceing $\chi_s^{\mp}(x)$ by $y^{-\frac{1}{2}}g_s^{\mp}(y)$, we arrive at an equation similar to Eqn. (62)

$$\left\{\left(\frac{d^2}{dy^2}-\frac{1}{4}+\frac{iaE}{y}+\frac{1/4-\left(ia\sqrt{E^2-m^2}\right)^2}{y^2}\right)I_{2^{2s}}\pm\frac{s(\sigma_3)^{2s}}{y}\right\}g_s^{\mp}(y)=0, \quad (56)$$

which also has for solutions

$$g_s^{\mp}(y)=\begin{pmatrix} C_3^{\mp}M_{\kappa_s^{\pm},-\mu}(y)+C_4^{\mp}M_{\kappa_s^{\pm},\mu}(y) \\ D_3^{\mp}M_{\kappa_s^{\pm},-\mu}(y)+D_4^{\mp}M_{\kappa_s^{\pm},\mu}(y) \end{pmatrix}. \quad (66)$$

Solutions of Eqns. (63) and (66) can be grouped together into a single equation including solutions relative to $x<0$ and $x>0$

$$\zeta_s^{\mp}(y)=f_s^{\mp}(y)\theta(-x)+g_s^{\mp}(y)\theta(x), \quad (67)$$

with $y=2iaV_0e^{-\frac{|x|}{a}}$ and $\theta(x)$ is the Heaviside function. The stationary solution of Eqn. (58) is finally

$$\Phi_s^c(x)=\begin{pmatrix}\Phi_s^-(x)\\ \Phi_s^+(x)\end{pmatrix}=y^{-\frac{1}{2}}\begin{pmatrix}\left[\frac{\sigma_3+\sigma_1}{2}\right]^{(2s)}\zeta_s^-(y)\\ \left[\frac{\sigma_3+\sigma_1}{2}\right]^{(2s)}\zeta_s^+(y)\end{pmatrix}. \quad (68)$$

In addition, constants $C_j^{\mp}$ and $D_j^{\mp}$ ($j=1\rightarrow 4$) are not completely independent but are interconnected by the continuity condition of the wave function and its first derivative in the vicinity of zero point $x=0$ or $y=2iaV_0=\lambda$. To determine these conditions, let us return to the KG-s Eqn. (58) and examined what happens near a point x_0 where the potential $V(x)$ has a jump of the form

$$V(x)=\begin{cases}V_1(x) & \text{for} \quad x<x_0\\ V_2(x) & \text{for} \quad x>x_0\end{cases} \quad (69)$$

with

$$\frac{dV(x)}{dx}=\begin{cases}\frac{dV_1(x)}{dx} & \text{for} \quad x<x_0\\ \left[V_2(x_0^+)-V_1(x_0^-)\right]\delta(x-x_0) & \text{for} \quad x=x_0\\ \frac{dV_2(x)}{dx} & \text{for} \quad x>x_0\end{cases} \quad (70)$$

$\delta(x-x_0)$ is the delta function. Let us integrate Eqn. (58) on the domain $\left[x_0^-,x_0^+\right]$ we get:

$$\Phi_s^{\mp,c}(x_0^+)=\Phi_s^{\mp,c}(x_0^-) \quad (71)$$

$$\frac{d\Phi_s^{\mp,c}(x_0^+)}{dx}=\frac{d\Phi_s^{\mp,c}(x_0^-)}{dx}\pm 2ise\left[V_2(x_0^+)-V_1(x_0^-)\right](\sigma_3)_s^{2s}\Phi_s^{\mp,c}(x_0) \quad (72)$$

By applying the continuity conditions given by Eqns. (71) and (72) to $x=0$, a simple calculation yields

$$C_1^{\mp}=C_2^{\mp}W_{12}^{\mp}+C_4^{\mp}W_{14}^{\mp},\ C_3^{\mp}=C_2^{\mp}W_{32}^{\mp}+C_4^{\mp}W_{34}^{\mp} \\ D_1^{\mp}=D_2^{\mp}W_{12}^{\pm}+D_4^{\mp}W_{14}^{\pm},\ D_3^{\mp}=D_2^{\mp}W_{32}^{\pm}+D_4^{\mp}W_{34}^{\pm} \quad (73)$$

Where, the coefficients are defined as follows:

$$W_{12}^{\mp}=W_{34}^{\pm}=-\frac{F_1^{\mp}(F_2^{\pm})'+F_2^{\pm}(F_1^{\mp})'}{\left(F_1^{\mp}F_1^{\pm}\right)'}, \\ W_{14}^{\mp}=W_{32}^{\pm}=\frac{F_2^{\mp}(F_1^{\mp})'-F_1^{\mp}(F_2^{\mp})'}{\left(F_1^{\mp}F_1^{\pm}\right)'}. \quad (74)$$

$F_1^{\mp}$, $F_2^{\mp}$, $\left(F_1^{\mp}\right)'$ and $\left(F_2^{\mp}\right)'$ are defined by

$$\left(F_1^{\mp}\right)=\left[M_{\kappa_s^{\pm},-\mu}(y)\right]_{y=\lambda}=y^{1/2-\mu}F_1(1/2+\mu,1+2\mu,y), \\ \left(F_2^{\mp}\right)=\left[M_{\kappa_s^{\pm},+\mu}(y)\right]_{y=\lambda}=y^{1/2+\mu}F_1(1/2-\mu,1-2\mu,y), \\ \left(F_1^{\mp}\right)'=\left(-\frac{1}{2}+\frac{\lambda}{2}-\kappa_s^{\pm}\right)M_{\kappa_s^{\pm},-\mu}(y)+\left(\frac{1}{2}-\mu+\kappa_s^{\pm}\right)_{y=\lambda}M_{\kappa_s^{\pm}+1,-\mu}(y), \\ \left(F_2^{\mp}\right)'=\left(-\frac{1}{2}+\frac{\lambda}{2}-\kappa_s^{\pm}\right)M_{\kappa_s^{\pm},+\mu}(y)+\left(\frac{1}{2}+\mu+\kappa_s^{\mp}\right)_{y=\lambda}M_{\kappa_s^{\pm}+1,+\mu}(y). \quad (75)$$

To calculate the scattering matrix $\hat{S}^c_{KG-s}$, we first look for the asymptotic behavior of the stationary wave function $\Phi_s^{\mp,c}(x)$ at very great distance $|x|\mapsto\infty$ or again for $y\rightarrow 0$. By virtue of the known formula ${}_1F_1(a,b,0)=1$, and $M_{\kappa_s^{\pm},\pm\mu}(y\rightarrow 0)\rightarrow y^{1/2\pm\mu}e^{-y/2}$ [20], the stationary wave function is written:

$$\Phi_s^{\mp,c}(x)=\begin{cases}\left[\frac{\sigma_3+\sigma_1}{2}\right]^{(2s)}\left[\lambda^{-\mu}\begin{pmatrix}C_3^{\mp}\\ D_3^{\mp}\end{pmatrix}e^{ikx}+\lambda^{\mu}\begin{pmatrix}C_4^{\mp}\\ D_4^{\mp}\end{pmatrix}e^{-ikx}\right] & x\rightarrow+\infty\\ \left[\frac{\sigma_3+\sigma_1}{2}\right]^{(2s)}\left[\lambda^{-\mu}\begin{pmatrix}C_1^{\mp}\\ D_1^{\mp}\end{pmatrix}e^{ikx}+\lambda^{\mu}\begin{pmatrix}C_2^{\mp}\\ D_2^{\mp}\end{pmatrix}e^{-ikx}\right] & x\rightarrow-\infty\end{cases} \quad (76)$$

Comparing only (76) with (19) we obtain:

$$\Phi_s^{\mp,c}(in,-\infty)=\lambda^{\mu}\begin{pmatrix}C_2^{\mp}+D_2^{\mp}\\ C_2^{\mp}-D_2^{\mp}\end{pmatrix},\ \Phi_s^{\mp,c}(in,+\infty)=\lambda^{\mu}\begin{pmatrix}C_4^{\mp}+D_4^{\mp}\\ C_4^{\mp}-D_4^{\mp}\end{pmatrix}, \\ \Phi_s^{\mp,c}(out,-\infty)=\lambda^{-\mu}\begin{pmatrix}C_1^{\mp}+D_1^{\mp}\\ C_1^{\mp}-D_1^{\mp}\end{pmatrix},\ \Phi_s^{\mp,c}(out,+\infty)=\lambda^{-\mu}\begin{pmatrix}C_3^{\mp}+D_3^{\mp}\\ C_3^{\mp}-D_3^{\mp}\end{pmatrix}. \quad (77)$$

Substituting $\Phi_s^{\mp,c}(out,\mp\infty)$ and $\Phi_s^{\mp,c}(in,\mp\infty)$ given by Eqn. (77) in Eqn. (21) and using Eqns. (73),(45) and

(46), we can determine $\hat{S}_{KG-s}$ for Cusp potential:

$$\begin{pmatrix} \Phi_s^{-,c}(out,+\infty) \\ \Phi_s^{+,c}(out,+\infty) \\ \Phi_s^{-,c}(out,-\infty) \\ \Phi_s^{+,c}(out,-\infty) \end{pmatrix} = \begin{pmatrix} T_s^{+,c} & 0 & R_s^{+,c} & 0 \\ 0 & T_s^{-,c} & 0 & R_s^{-,c} \\ R_s^{-,c} & 0 & T_s^{-,c} & 0 \\ 0 & R_s^{+,c} & 0 & T_s^{+,c} \end{pmatrix} \begin{pmatrix} \Phi_s^{-,c}(in,-\infty) \\ \Phi_s^{+,c}(in,-\infty) \\ \Phi_s^{-,c}(in,+\infty) \\ \Phi_s^{+,c}(in,+\infty) \end{pmatrix} \quad (78)$$

Where,

$$\left. \begin{array}{l} T_s^{+,c} = \begin{pmatrix} t_s^{-,c} & t_s^{+,c} \\ t_s^{+,c} & t_s^{-,c} \end{pmatrix},\ T_s^{-,c} = \begin{pmatrix} t_s^{-,c} & -t_s^{+,c} \\ -t_s^{+,c} & t_s^{-,c} \end{pmatrix}, \\ t_s^{\mp,c} = \frac{1}{2}\lambda^{-2\mu}\left(W_{14}^{+} \pm W_{14}^{-}\right), \\ R_s^{+,c} = \begin{pmatrix} r_s^{-,c} & r_s^{+,c} \\ r_s^{+,c} & r_s^{-,c} \end{pmatrix},\ R_s^{-,c} = \begin{pmatrix} r_s^{-,c} & -r_s^{+,c} \\ -r_s^{+,c} & r_s^{-,c} \end{pmatrix}, \\ r_s^{\mp,c} = \frac{1}{2}\lambda^{-2\mu}\left(W_{12}^{+} \pm W_{12}^{-}\right). \end{array} \right\} \quad (79)$$

$t_s^{\mp,c}$ and $r_s^{\mp,c}$ are functions of E, V_0 and m.

Finally, let us give the expression of the matrix M_{KG-s}^{c}, of the phase shift $\left(\delta_s^{l,c}\right)$, of the wave function and the transmission and reflection coefficients in the case of Cups potential. Using the reasoning previously exposed for the treatment of a symmetrical potential $V(x) = V(-x)$. for matrix M_{KG-s}, Eqn. (50) gives

$$M_{KG-s}^{c} = I_{2^s} \otimes \begin{pmatrix} \exp\left(i\delta_s^{0,c}\right) & 0 \\ 0 & \exp\left(i\delta_s^{1,c}\right) \end{pmatrix}, \quad (80)$$

Where,

$$\exp\left(i\delta_s^{l,c}\right) = t_s^{-,c} + (-1)^l \sqrt{\left(t_s^{+,c}\right)^2 + \left(r_s^{-,c}\right)^2 - \left(r_s^{+,c}\right)^2}. \quad (81)$$

To calculate the wave function, substituting Eqn. (81) into Eqn. (55), the result into Eqn. (54), we find

$$\Phi_{s,L}^{\mp,c}(x) = \begin{cases} \sqrt{2}\chi_s^{\mp}(in,-\infty)\left[e^{ikx} + f_s^{-,c}e^{-ikx}\right] & x \to -\infty \\ \sqrt{2}\chi_s^{\mp}(in,-\infty)\left[e^{ikx} + \left(1 + f_s^{+,c}\right)e^{ikx}\right] & x \to +\infty \end{cases} \quad (82)$$

Here,

$$f_s^{+,c} = \frac{1}{2}\left(e^{2i\delta_s^{0,c}} + e^{2i\delta_s^{1,c}}\right) - 1 = \lambda^{-2\mu}W_{32}^{+} - 1,\ f_s^{-,c} = \frac{1}{2}\left(e^{2i\delta_s^{0,c}} - e^{2i\delta_s^{1,c}}\right) = \lambda^{-2\mu}W_{12}^{-}. \quad (83)$$

From Eqn. (56), the reflection and transmission coefficients are given by

$$\begin{aligned} \mathsf{T}_s^{c} &= |1 + f_s^{+,c}|^2 = \tfrac{1}{4}\left|e^{2i\delta_s^{0,c}} + e^{2i\delta_s^{1,c}}\right|^2 = \left|\lambda^{-2\mu}\right|^2 \left|W_{32}^{-}\left(W_{32}^{-}\right)^*\right|, \\ \mathsf{R}_s^{c} &= | f_s^{-,c}|^2 = \tfrac{1}{4}\left|e^{2i\delta_s^{0,c}} - e^{2i\delta_s^{1,c}}\right|^2 = \left|\lambda^{-2\mu}\right|^2 \left|W_{12}^{-}\left(W_{12}^{-}\right)^*\right|. \end{aligned} \quad (84)$$

5. Results and discussion

The resolution of KG-s (Eqn. (60)) interacting with the Dirac delta potential $V(x) = \alpha\delta(x)$ (α being a positive parameter), leads to infinite matrices. This comes from the term $V^2(x)$ in (60). This term gives a square delta $\delta(x)\delta(x) = \delta(0)\delta(x) \to \infty$. For this reason the delta potential is replaced by a regular potential. This potential, we choose it as that of Cusp. By assimilation of α to greatness $\alpha = \int_{-\infty}^{+\infty} V_0 \exp(-\frac{|x|}{a})dx = 2aV_0$, going to the limit $a \to 0$, $V_0 \to \infty$, the potential of Cusp becomes the delta potential $\lim_{a\to 0^+, V_0 \to \infty} V_0 \exp(-\frac{|x|}{a}) \to \alpha\delta(x)$ [10].

By a passage to the limit $a \to 0$ and $V_0 \to \infty$, we can express the wave function, the scattering matrix, the transmission and reflection coefficients and the phase shifts for a particle of KG-s interacting with the delta potential. For the coefficients R_s and T_s,

$$\mathsf{R}_s^{\delta} = \lim_{a\to 0^+, V_0\to\infty} \mathsf{R}_s^{c} \to 1,\ \mathsf{T}_s^{\delta} = \lim_{a\to 0^+, V_0\to\infty} \mathsf{T}_s^{c} \to 0.$$

Similarly, it is easy to deduce the scattering matrix $\hat{S}_D$ corresponding to Eqn. (2), which connects the outgoing wave $\Psi_{1/2}(out)$ to the incident wave $\Psi_{1/2}(in)$, by the relation $\Psi_{1/2}(out) = \hat{S}_D \Psi_{1/2}(in)$. To calculate $\hat{S}_D$, we start by noticing that for $|x| \to \infty$, $A\!\!\!/ \to 0$ and $p\!\!\!/ - eA\!\!\!/ + m \to p\!\!\!/ + m$. Inserting this into (2), we derive $\Phi_{1/2}(out) = \frac{1}{p\!\!\!/+m}\Psi_{1/2}(out)$ and $\Phi_{1/2}(in) = \frac{1}{p\!\!\!/+m}\Psi_{1/2}(in)$. Substituting these into Eqn. (23) and by a straight forward calculation one obtain $\hat{S}_D = \hat{S}_{KG-1/2}$.

6. Conclusion

Our work has been organized around two major parts. In the first part, we considered the spin-s $(s = 0, 1/2)$ Klien Gordon equation interacting with the one-dimensional symmetrical scalar potential. The formalism of the scattering matrix $\hat{S}_{KG-s}$ for a spin-s related particle interacting with a localized electromagnetic field was first constructed that allowed us to retrieve reflection and transmission coefficients again. In the second part, as an application of the results, the case of the Cusp's potential was investigated in detail and we solved the KG equation for a spin-s relativistic particle. The solution was given in analytical form using Whittaker's functions. From the asymptotic behaviour and by the conditions of continuity we drew the elements of scattering matrix $\hat{S}_{KG-s}^{c}$, phase shifts, reflection R_s^{c} and the transmission T_s^{c} coefficients.

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