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Modified Schrödinger Equation with Modified QM Gravitational and Harmonic Oscillator Potentials in Symmetries of Non-commutative QM

Abdelmadjid Maireche

Laboratory of Physics and Material Chemistry, Physics Department, Faculty of Sciences, University of M'sila, M'sila, Algeria. Email: Abdelmadjid.maireche@univ-msila.dz

Modified Schrodinger equation plays a fundamental role for describing the behavior of a particle at microscopic scale. In this context, we have studied the non-relativistic energy spectra of the modified quantum mechanical gravitational potential plus the harmonic oscillator (MQMGHO) potential using the generalized Bopp's shift method and standard perturbation theory within the framework of non-commutative 3-dimensional phase space (NC: 3D-RSP) symmetries. Furthermore, we have shown that the modified Hamiltonian operator containing a perturbed Coulomb potential, which physically means the global Hamiltonian of MQMGHO potential, can be described as Hydrogenic atoms interacting with strong potential composed of quantum mechanical gravitational potential plus the harmonic oscillator (QMGHO) potential and an auxiliary part. The bound state energy eigenvalues, in terms of discreet atomic quantum numbers (j , (n,l) and m), four infinitesimal parameters $(\Theta,\chi,\overline{\theta},\overline{\sigma})$ induced by position-position and phase-phase non-commutativity, in addition to the dimensional parameters of MQMGHO potential and the corresponding non-commutative Hamiltonian operator were obtained for Hydrogenic atoms. Moreover, we have shown that, the total complete degeneracy of energy levels of studied potential were changed and replaced by new values $2n^2$.

1. Introduction

Recently, there has been a great interest to study physical and chemical phenomena in the generalized quantum mechanics (non-commutative quantum mechanics (NCQM)). There has been an increasing interest in finding the analytical solutions that play a crucial role for getting complete physical and chemical information about quantum mechanical systems. In the non-relativistic case the exact bound state solutions of the modified Schrödinger equation (MSE) are only possible for some potentials of chemical and physical interest. In the symmetries of NCQM, which was made known firstly by Heisenberg and was formalized by Snyder at 1947, was suggested by recent results in string theory [1-5]. In this work, our aim is to solve the MSE for the modified quantum mechanical gravitational (MQMG) potential plus the modified harmonic oscillator (MHO) potential via the generalized Bopp's shift method and standard perturbation theory in the case of the symmetries of NCQM. The MQMGHO potential takes the form:

$$V_{mgo}(r)=V_0+(\beta-\alpha V_0)r+\left(\alpha^2V_0+\frac{1}{2}\mu\omega^2\right)r^2\rightarrow$$
$$V_{mgo}(\hat{r})=V_{mgo}(r)-\left(\alpha^2V_0+\frac{1}{2}\mu\omega^2+\frac{\beta-\alpha V_0}{2r}\right)\vec{\mathbf{L}}\vec{\Theta}+\frac{\vec{\mathbf{L}}\vec{\overline{\theta}}}{2\mu} \quad (1)$$

In 2005, B. Santos *et al.* [6] have studied the motion of a particle in a gravitational field using the QMGP without the exponential term and in 2014, Ita *et al* [7] have applied the NU method to
QMGHO potential, where they obtained bound state s-wave solution of the SE equation. Recently, H. Louis et *al.* [8] solved the SE for QMGHO potential by WKB approximation method. It is known that the QMGP could be used to calculate the energy of a body falling under gravity from the quantum mechanical point of view [7-8]. The modified harmonic oscillator potential plays a basic role in chemical and molecular physics. We want to extend, this study [7-8] to the case of NCQM symmetries. The NCQM structure based to NC canonical commutations relations in both Schrödinger and Heisenberg pictures (SP and HP), respectively, as follows (Throughout this paper, the natural units $c=\hbar=1$ will be used) [9-23]:

$$\begin{cases}[x_i,p_j]=[x_i(t),p_j(t)]=i\delta_{ij}\\ [x_i,x_j]=[x_i(t),x_j(t)]=0\\ [p_i,p_j]=[p_i(t),p_j(t)]=0\end{cases}\Rightarrow$$
$$\begin{cases}\left[\hat{x}_i\overset{*}{,}\hat{p}_j\right]=\left[\hat{x}_i(t)\overset{*}{,}\hat{p}_j(t)\right]=i\hbar_{eff}\delta_{ij}\Rightarrow\Delta\hat{x}_i\Delta\hat{p}_j\equiv\Delta\hat{x}_i(t)\Delta\hat{p}_j(t)\geq\frac{\delta_{ij}}{2}\\ \left[\hat{x}_i\overset{*}{,}\hat{x}_j\right]=\left[\hat{x}_i(t)\overset{*}{,}\hat{x}_j(t)\right]=i\theta_{ij}\Rightarrow\Delta\hat{x}_i\hat{x}_j\equiv\Delta\hat{x}_i(t)\hat{x}_j(t)\geq\frac{|\theta_{ij}|}{2}\\ \left[\hat{p}_i\overset{*}{,}\hat{p}_j\right]=\left[\hat{p}_i(t)\overset{*}{,}\hat{p}_j(t)\right]=i\overline{\theta}_{ij}\Rightarrow\Delta\hat{p}_i\Delta\hat{p}_j\equiv\Delta\hat{p}_i(t)\Delta\hat{p}_j(t)\geq\frac{|\overline{\theta}_{ij}|}{2}\end{cases} \quad (2)$$

Where, the non-commutativity parameters θ_{ij} and $\overline{\theta}_{ij}$ are real-values and elements of the antisymmetric constant matrix with a dimension of $[x_i]^2$ and $[p_i]^2$, respectively and $\hbar_{eff}$ is the effective Planck constant. However, the new operators $\hat{\xi}(t)\equiv(\hat{x}_I\vee\hat{p}_i)(t)$ in HP are depending to the corresponding new operators $\hat{\xi}=\hat{x}_i\vee\hat{p}_i$ in SP from the following projections relations:

$$\xi(t)=\exp(i\hat{H}_{mgo}(t-t_0))\xi\exp(-i\hat{H}_{mgo}(t-t_0))\Rightarrow$$

$$\hat{\xi}(t)=\exp(i\hat{H}_{nc-mgo}(t-t_0))*\hat{\xi}*\exp(-i\hat{H}_{nc-mgo}(t-t_0)) \quad (3)$$

The dynamics of new systems $\frac{d\xi(t)}{dt}$ are described from the following motion equations in NCQM

$$\frac{d\xi(t)}{dt}=\left[\xi(t),\hat{H}_{mgo}\right]\Rightarrow$$
$$\frac{d\hat{\xi}(t)}{dt}=\left[\hat{\xi}(t)\overset{*}{,}\hat{H}_{nc-mgo}\right]\equiv\hat{\xi}(t)*\hat{H}_{nc-mgo}-\hat{H}_{nc-mgo}*\hat{\xi}(t) \quad (4)$$

The two operators $\hat{H}_{mgo}$ and $\hat{H}_{nc-mgo}$ represent the ordinary and quantum Hamiltonian operators for QMGHO potential and MQMGHO potential in the QM and NCQM, respectively. The very small two parameters $\theta^{\mu\nu}$ and $\overline{\theta}^{\mu\nu}$ (compared to the energy) are elements of two antisymmetric real matrixes and $(*)$ denote to the new star product, which is generalized between two arbitrary functions $(fg)(x,p)$ to the new form $\hat{f}(\hat{x},\hat{p})\hat{g}(\hat{x},\hat{p})\equiv(f*g)(x,p)$ in ordinary 3-dimensional space-phase [17-23]:

$$(f,g)(x,p)\rightarrow(f*g)(x,p)$$
$$=\left(fg-\frac{i}{2}\theta^{\mu\nu}\partial_\mu^x f\,\partial_\nu^x g-\frac{i}{2}\overline{\theta}^{\mu\nu}\partial_\mu^p f\,\partial_\nu^p g\right)(x,p) \quad (5)$$

Where, $\left(\partial_\mu^x,\partial_\mu^p\right)=\left(\frac{\partial}{\partial x^\mu},\frac{\partial}{\partial p^\mu}\right)$. The effects of (space-space) and (phase-phase) noncommutativity properties, respectively induce the second and the third terms in the above equation. The organization scheme of the recent work is given as follows. The ordinary SE with QGMHO potential will be reviewed in Sec. 2 based on [7-8]. Sec.3 is devoted to studying the MSE by applying the generalized Bopp's shift method for MQGHO potential. In the next subsection, by applying standard perturbation theory to find the quantum spectrum of n^{th} excited levels for spin-orbital interaction in the framework of the global group (NC-3D: RSP) and then, we derive the magnetic spectrum for MQMGHOP. In the Sec. 4, we resume the global spectrum and corresponding NC Hamiltonian operator for MQMGHO potential. Finally, the short concluding remarks have been presented in the Sec.5.

2. Overview of the eigenfunctions and the energy eigenvalues for QMGHO potential

In this section, we are going to study some basic properties of the time-independent Schrödinger equation for a quantum mechanical gravitational potential plus the harmonic oscillator potential (QMGHO) of the form [7-8]:

$$V(z)=mgz+\delta e^{-kz}+\frac{1}{2}\mu\omega^2z^2\rightarrow$$
$$V_{mgo}(r)=mgr+\delta e^{-kr}+\frac{1}{2}\mu\omega^2r^2\cong V_0+(\beta-\alpha V_0)r+\left(\alpha^2V_0+\frac{1}{2}\mu\omega^2\right)r^2 \quad (6)$$

Where, z is the displacement, k is momentum, m is the mass, g is gravitational acceleration, δ is an adjustable parameter, μ is the reduced mass, ω is the angular frequency, and $\beta=mg$, $\alpha=k$ and $\delta=V_0$. If we insert this potential into the time-independent SE, the radial part reads [7]

$$\frac{d^2R_{nl}(r)}{dr^2}+2\mu\left[\begin{array}{l}E_{nl}-V_0-(\beta-\alpha V_0)r-\\-\left(\alpha^2V_0+\frac{1}{2}\mu\omega^2\right)r^2-\frac{l(l+1)}{r^2}\end{array}\right]R_{nl}(r)=0 \quad (7)$$

The radial parts of the wave functions are shown as a function of the Laguerre polynomial in terms of some parameters [7] as

$$R_{nl}(r)=Nr^{(1+\varepsilon)/2}\exp\left(-\sqrt{\gamma_1}r\right)L_n^\varepsilon\left(\sqrt{\gamma_1}r\right) \quad (8)$$

Where, $\varepsilon=2\sqrt{1/4+\gamma_3}$, $\gamma_3=-2\mu(E-V_0)$ and $\gamma_1=2\mu\left(\alpha^2V_0+\frac{1}{2}\mu\omega^2\right)$, therefore, the complete wave function $\Psi(r,\theta,\varphi)$ and the energy E_n of the potential in Eqn. (6) are given by

$$\Psi(r,\theta,\varphi)=N\frac{r^{(1+\varepsilon)/2}}{r}\exp\left(-\sqrt{\gamma_1}r\right)L_n^\varepsilon\left(\sqrt{\gamma_1}r\right)Y_l^m(\theta,\varphi) \quad (9)$$

and [7-8] by

$$E_n=V_0+\frac{1}{2\mu}\left\{\begin{array}{l}(n+1/2)\left[(n+1/2)+\frac{2\mu(\beta-\alpha V_0)}{\sqrt{2\mu(\alpha^2V_0+\mu\omega^2/2)}}\right]\\+\frac{2\mu(\beta-\alpha V_0)^2}{4\left[2\mu(\alpha^2V_0+\mu\omega^2/2)\right]}\end{array}\right\} \quad (10)$$

SOLUTION OF THE MODIFIED SCHRÖDINGER EQUATION

2.1 Review of generalized Bopp's shift method

In this sub-section, we shall give an overview or a brief preliminary for MQMGHO potential in (NC: 3D-RSP) symmetries. To perform this task the physical form of MSE, it is necessary to replace ordinary three-dimensional Hamiltonian operators $\hat{H}(p_i, x_i)$, ordinary complex wave function $\Psi\left(\vec{r}\right)$ and ordinary energy E_{nl} by new three Hamiltonian operators $\hat{H}_{nc-mgo}(\hat{p}_i, \hat{x}_i)$, new complex wave function $\hat{\Psi}\left(\vec{\hat{r}}\right)$ and new values E_{nc-mgo}, respectively. In addition to replace the ordinary old product by the Moyal–Weyl product ($*$- product), which allows us to construct the MSE in (NC-3D: RSP) symmetries [18-26] as

$$\hat{H}_{mgo}\left(p_i, x_i\right)\Psi\left(\vec{r}\right) = E_{nl}\Psi\left(\vec{r}\right) \Rightarrow \hat{H}\left(\hat{p}_i, \hat{x}_i\right) * \Psi\left(\vec{\hat{r}}\right) = E_{nc-mgo}\Psi\left(\vec{\hat{r}}\right) \quad (11)$$

The Bopp's shift method employed in the solutions enables us to explore an effective way of obtaining the modified potential in NCQM, which is based on the following new commutators [24-34]:

$$\left[\hat{x}_i, \hat{x}_j\right] = \left[\hat{x}_i(t), \hat{x}_j(t)\right] = i\theta_{ij} \text{ and} \left[\hat{p}_i, \hat{p}_j\right] = \left[\hat{p}_i(t), \hat{p}_j(t)\right] = i\bar{\theta}_{ij} \quad (12)$$

It is well known that, in (NC: 3D-RSP), the new generalized positions and momentum coordinates $(\hat{x}_i, \hat{p}_i)$ can be obtained by the carrying out non-minimal substitution [19-24] as

$$\left(x_i, p_i\right) \Rightarrow \left(\hat{x}_i, \hat{p}_i\right) = \left(x_i - \frac{\theta_{ij}}{2} p_j, p_i + \frac{\bar{\theta}_{ij}}{2} x_j\right) \quad (13)$$

Where, (x_i, p_i) are corresponding usual generalized positions and momentum coordinates in CQM obeys the usual commutation relations $\left[x_i, p_j\right] = \left[x_i(t), p_j(t)\right] = i\delta_{ij}$. The above equation allows us to obtain the two operators $\hat{r}^2$ and $\hat{p}^2$ in (NC-3D: RSP) [25-30] as

$$\left(r^2, p^2\right) \Rightarrow \left(\hat{r}^2, \hat{p}^2\right) = \left(r^2 - \mathbf{L}\vec{\Theta}, p^2 + \vec{\mathbf{L}}\vec{\bar{\theta}}\right) \quad (14)$$

The two couplings $\mathbf{L}\Theta$ and $\vec{\mathbf{L}}\vec{\bar{\theta}}$ are $\left(L_x\Theta_{12} + L_y\Theta_{23} + L_z\Theta_{13}\right)$ and $\left(L_x\bar{\theta}_{12} + L_y\bar{\theta}_{23} + L_z\bar{\theta}_{13}\right)$, respectively and ($L_x$, L_y and L_z) are the three components of angular momentum operator $\vec{L}$ while $\Theta_{ij} = \theta_{ij}/2$. Thus, the reduced SE (without star product) can be written as

$$\begin{aligned} &\hat{H}\left(\hat{p}_i, \hat{x}_i\right) * \Psi\left(\vec{\hat{r}}\right) = E_{nc-mgo}\Psi\left(\vec{\hat{r}}\right) \\ &\Rightarrow H\left(\hat{p}_i, \hat{x}_i\right)\psi\left(\vec{r}\right) = E_{nc-mgo}\psi\left(\vec{r}\right) \end{aligned} \quad (15)$$

The new operator of Hamiltonian $H_{nc-mgo}(\hat{p}_i, \hat{x}_i)$ can be expressed as

$$H_{mgo}\left(p_i, x_i\right) \Rightarrow H_{nc-mgo}\left(\hat{p}_i, \hat{x}_i\right) \equiv H\left(\hat{x}_i = x_i - \frac{\theta_{ij}}{2} p_j, \hat{p}_i = p_i + \frac{\bar{\theta}_{ij}}{2} x_j\right) \quad (16)$$

Now, we want to find the MQMGHO potential $V_{mgo}(\hat{r})$ which can found from the relation given below as

$$V_{mgo}(r) \Rightarrow V_{mgo}(\hat{r}) = V_0 + \left(\beta - \alpha V_0\right)\hat{r} + \left(\alpha^2 V_0 + \frac{1}{2}\mu\omega^2\right)\hat{r}^2 \quad (17)$$

After straightforward calculations, we can obtain the important terms $(\beta - \alpha V_0)\hat{r}$ and $\left(\alpha^2 V_0 + \frac{1}{2}\mu\omega^2\right)\hat{r}^2$, which will be used to determine the MQMGHO potential in (NC: 3D- RSP) symmetries as:

$$\begin{aligned} &(\beta - \alpha V_0) r \rightarrow (\beta - \alpha V_0)\hat{r} = (\beta - \alpha V_0) r - \frac{\beta - \alpha V_0}{2r}\mathbf{L}\vec{\Theta} \\ &\left(\alpha^2 V_0 + \frac{1}{2}\mu\omega^2\right)\left\{r^2 \rightarrow \hat{r}^2 = r^2 - \mathbf{L}\vec{\Theta}\right\} \end{aligned} \quad (18)$$

By making the substitution above equation into Eqn. (17), we find the global our working new Hamiltonian operator $H_{\text{nc-}mgo}(\hat{r})$ satisfies the equation in (NC: 3D-RSP) symmetries:

$$H_{mgo}(p, x) \Rightarrow H_{nc-mgo}(\hat{r}) = H_{mgo}(p, x) + H_{\text{per-}mgo}(r) \quad (19)$$

Where, the operator $H_{mgo}(p, x)$ is just the ordinary Hamiltonian operator for MGHO potential in commutative space:

$$H_{mgo}(p, x) = \frac{p^2}{2\mu} + V_0 + (\beta - \alpha V_0) r + \left(\alpha^2 V_0 + \frac{1}{2}\mu\omega^2\right) r^2 \quad (20)$$

Whereas, the rest part $H_{\text{per-}mgo}(r)$ is proportional with two infinitesimals parameters (Θ and $\bar{\theta}$):

$$H_{\text{per-}mgo}(r) = -f(r)\mathbf{L}\vec{\Theta} + \frac{\vec{\mathbf{L}}\vec{\bar{\theta}}}{2\mu} \quad (21)$$

With, $f(r) \equiv \alpha^2 V_0 + \frac{1}{2}\mu\omega^2 + \frac{\beta - \alpha V_0}{2r}$ The above relationship clearly shows us the additive part $H_{\text{per-}mgo}(r)$ of the Hamiltonian operator containing a perturbed Columbic potential $\frac{\beta-\alpha V_0}{2r}\vec{\mathbf{L}}\vec{\Theta}$. This means physically that the global Hamiltonian $H_{nc-mgo}(\hat{r})$ can be describe as Hydrogenic atom interacting with strong potential composed with quantum mechanical gravitational potential plus the harmonic oscillator potential and an auxiliary part $-\left(\alpha^2 V_0+\frac{1}{2}\mu\omega^2\right)\vec{\mathbf{L}}\vec{\Theta}$. Thus, we can consider $H_{\text{per-}mgo}(r)$ as a perturbation terms compared with the principal Hamiltonian operator $H_{mgo}(p,x)$ in (NC: 3D-RSP) symmetries.

3.2 The modified spin-orbit spectrum for MQMGHO potential in (NC: 3D- RSP) symmetries

In this sub-section, we want to see the physical contribution of the generated Hamiltonian operator $H_{\text{per-}mgo}(r)$ and its effect on the principal energy E_n. To achieve this important objective, as a first step, we follow the same strategy that we saw in our previous works [31-38]; under such particular choice, one can easily reproduce both couplings ($\vec{\mathbf{L}}\vec{\Theta}$ and $\vec{\mathbf{L}}\vec{\vec{\theta}}$) to the new physical forms ($\gamma\Theta\vec{L}\vec{S}$ and $\gamma\bar{\theta}\vec{L}\vec{S}$), respectively. We have oriented the two-arbitrary vectors $\vec{\Theta}$ and $\vec{\vec{\theta}}$ to same direction of spin $\vec{S}$ (we have chosen the two vectors $\vec{\Theta}$ and $\vec{\vec{\theta}}$ parallel to the spin $\vec{S}$ of Hydrogenic atoms). Thus, the new forms of $H_{\text{per-}mgo}(r)$ for MQMGHO potential as follows

$$H_{\text{per-}mgo}(r)\equiv H_{\text{so-}mg}(r,\Theta,\bar{\theta})\equiv\gamma\left\{-\left(\alpha^2V_0+\frac{1}{2}\mu\omega^2+\frac{\beta-\alpha V_0}{2r}\right)+\frac{\bar{\theta}}{2\mu}\right\}\vec{L}\vec{S} \quad (22)$$

Here $\gamma\approx\frac{1}{137}$ is a new constant, which plays the role of the fine structure constant. Furthermore, the above perturbative terms $H_{\text{per-}mgo}(r)$ can be rewritten to the following new form as

$$H_{so-mgo}(r,\Theta,\bar{\theta})=\frac{\gamma}{2}\left\{-\left(\alpha^2V_0+\frac{1}{2}\mu\omega^2+\frac{\beta-\alpha V_0}{2r}\right)\Theta+\frac{\bar{\theta}}{2\mu}\right\}G^2 \quad (23)$$

Where, $G^2\equiv\vec{J}^2-\vec{L}^2-\vec{S}^2$. This operator relates the coupling between spin $\vec{S}$ and orbital momentum $\vec{L}\vec{S}$. The set ($H_{so-mgo}(r,\Theta,\bar{\theta})$, J^2, L^2, S^2 and J_z) forms a complete set of conserved physics quantities and for $\overleftrightarrow{S}=\vec{1/2}$, the eigenvalues of the spin orbit coupling operator are $k_\pm\equiv\frac{1}{2}\left\{\left(l\pm\frac{1}{2}\right)(l\pm\frac{1}{2}+1)+l(l+1)-\frac{3}{4}\right\}$ corresponding to $j=l+1/2$ (spin up) and $j=l-1/2$ (spin down), respectively, then, one can form a diagonal (3×3) matrix, with diagonal elements are $(H_{so-mgo})_{11}$, $(H_{so-mgo})_{22}$ and $(H_{so-mgo})_{33}$ for MQMGHO potential in (NC: 3D-RSP) symmetries as

$$\begin{aligned}(H_{so-mgo})_{11}&=\gamma k_+\left(-\left(\alpha^2V_0+\frac{1}{2}\mu\omega^2+\frac{\beta-\alpha V_0}{2r}\right)\Theta+\frac{\bar{\theta}}{2\mu}\right) \text{ if } j=l+1/2\\ (H_{so-mgo})_{22}&=\gamma k_-\left(-\left(\alpha^2V_0+\frac{1}{2}\mu\omega^2+\frac{\beta-\alpha V_0}{2r}\right)\Theta+\frac{\bar{\theta}}{2\mu}\right) \text{ if } j=l-1/2\\ (H_{so-mgo})_{33}&=0\end{aligned} \quad (24)$$

After a detailed calculation one can show that the new radial function $R_{nl}(r)$ satisfying the following differential equation for MQMGHO potential:

$$\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{l(l+1)}{r^2}\right)R_{nl}(r)+2\mu\left[E_{nl}-V_0-(\beta-\alpha V_0)r-\left(\alpha^2V_0+\frac{1}{2}\mu\omega^2\right)r^2+\left(\alpha^2V_0+\frac{1}{2}\mu\omega^2+\frac{\beta-\alpha V_0}{2r}\right)\vec{\mathbf{L}}\vec{\Theta}-\frac{\vec{\mathbf{L}}\vec{\vec{\theta}}}{2\mu}\right]R_{nl}(r)=0 \quad (25)$$

The two terms which composed the expression of $H_{\text{per-}mgo}(r)$ are proportional to two infinitesimals parameters (Θ and $\bar{\theta}$). Thus, in what follows, we proceed to solve the modified radial part of the MSE that is, Eqn. (24) by applying standard perturbation theory for their exact solutions at first order of two parameters Θ and $\bar{\theta}$. After this sub-section, we study the fundamentally rich systems of (NC: 3D- RSP), which will be used to generate the energy of studied potential.

3.3 The exact modified spin-orbital spectrum for MQMGHO potential in global (NC: 3D- RSP) symmetries

The purpose here is to give a complete prescription for determine the energy level of n^{th} excited states, of Hydrogenic atoms with MQMGHO potential, we first find the corrections

$E_{\text{u-}mgo}$ and $E_{\text{d-}mgo}$ which have $j = l + 1/2$ (spin up) and $j = l - 1/2$ (spin down), respectively, at first order of two parameters Θ and $\bar{\theta}$ obtained by applying the standard perturbation theory to find the following:

$$\begin{aligned} E_{u-mgo} &= -\gamma N^2 k_+ \int_0^{+\infty} r^{1+\varepsilon} \exp\left(-2\sqrt{\gamma_1}r\right)\left[L_n^{\varepsilon}\left(2\sqrt{\gamma_1}r\right)\right]^2 \left(\left(\alpha^2 V_0 + \frac{1}{2}\mu\omega^2 + \frac{\beta - \alpha V_0}{2r}\right)\Theta - \frac{\bar{\theta}}{2\mu}\right) dr \\ E_{d-mgo} &= -\gamma N^2 k_- \int_0^{+\infty} r^{1+\varepsilon} \exp\left(-2\sqrt{\gamma_1}r\right)\left[L_n^{\varepsilon}\left(2\sqrt{\gamma_1}r\right)\right]^2 \left(\left(\alpha^2 V_0 + \frac{1}{2}\mu\omega^2 + \frac{\beta - \alpha V_0}{2r}\right)\Theta - \frac{\bar{\theta}}{2\mu}\right) dr \end{aligned} \tag{26}$$

Now, we can write the above two equations to the new form:

$$\begin{aligned} E_{u-mgo}(n,\varepsilon,\gamma_1,l) &= -\gamma N^2 k_+ \left\{\Theta T_1(n,\varepsilon,\gamma_1) + \Theta T_2(n,\varepsilon,\gamma_1) - \frac{\bar{\theta}}{2\mu} T_3(n,\varepsilon,\gamma_1)\right\} \\ E_{d-mgo}(n,\varepsilon,\gamma_1,l) &= -\gamma N^2 k_- \left\{\Theta T_1(n,\varepsilon,\gamma_1) + \Theta T_2(n,\varepsilon,\gamma_1) - \frac{\bar{\theta}}{2\mu} T_3(n,\varepsilon,\gamma_1)\right\} \end{aligned} \tag{27}$$

Moreover, the expressions of the three factors $T_1(n,\varepsilon,\gamma_1)$, $T_2(n,\varepsilon,\gamma_1)$ and $T_3(n,\varepsilon,\gamma_1)$ are given by:

$$\begin{aligned} T_1(n,\varepsilon,\gamma_1) &= \left(\alpha^2 V_0 + \frac{1}{2}\mu\omega^2\right)\int_0^{+\infty} r^{2+\varepsilon-1} \exp\left(-2\sqrt{\gamma_1}r\right)\left[L_n^{\varepsilon}\left(2\sqrt{\gamma_1}r\right)\right]^2 dr = \left(\alpha^2 V_0 + \frac{1}{2}\mu\omega^2\right) T_3(n,\varepsilon,\gamma_1) \\ T_2(n,\varepsilon,\gamma_1) &= \frac{\beta - \alpha V_0}{2}\int_0^{+\infty} r^{1+\varepsilon-1} \exp\left(-2\sqrt{\gamma_1}r\right)\left[L_n^{\varepsilon}\left(2\sqrt{\gamma_1}r\right)\right]^2 dr \end{aligned} \tag{28}$$

For the ground state, we have $L_{n=0}^{\varepsilon}\left(2\sqrt{\gamma_1}r\right) = 1$. It is convenient to apply the following special integral [39]:

$$\int_0^{+\infty} x^{\nu-1} \exp\left(-\mu x^p\right) dx = \frac{1}{p}\mu^{-\frac{\nu}{p}}\Gamma\left(\frac{\nu}{p}\right) \tag{29}$$

With conditions ($\operatorname{Re}\mu \rangle 0$, $\operatorname{Re}\nu \rangle 0$ and $p \rangle 0$) and $\Gamma\left(\frac{\nu}{p}\right)$ the ordinary Gamma function. After straightforward calculations, we can obtain the explicit results as follows

$$\begin{aligned} T_1(n=0,\varepsilon,\gamma_1) &= \left(\alpha^2 V_0 + \frac{1}{2}\mu\omega^2\right)(4\gamma_1)^{-(2+\varepsilon)/2}\Gamma(2+\varepsilon) \\ &= \left(\alpha^2 V_0 + \frac{1}{2}\mu\omega^2\right) T_3(n=0,\varepsilon,\gamma_1) \\ T_2(n=0,\varepsilon,\gamma_1) &= \frac{\beta - \alpha V_0}{2}\frac{1}{p}(4\gamma_1)^{-(1+\varepsilon)/2}\Gamma(1+\varepsilon) \end{aligned} \tag{30}$$

For the first excited state, we have $L_{n=1}^{\varepsilon}\left(2\sqrt{\gamma_1}r\right) = -2\sqrt{\gamma_1}r + \varepsilon + 1$. Thus, the three factors $T_1(n=1,\varepsilon,\gamma_1)$, $T_2(n=1,\varepsilon,\gamma_1)$ and $T_3(n=1,\varepsilon,\gamma_1)$ are given by

$$\begin{aligned} T_1(n=1,\varepsilon,\gamma_1) &= \left(\alpha^2 V_0 + \frac{1}{2}\mu\omega^2\right) T_3(n=1,\varepsilon,\gamma_1) \\ &= \left(\alpha^2 V_0 + \frac{1}{2}\mu\omega^2\right)\left\{4\gamma_1 \int_0^{+\infty} r^{4+\varepsilon-1}\exp\left(-2\sqrt{\gamma_1}r\right)dr - 2\sqrt{\gamma_1}(\varepsilon+2)\int_0^{+\infty} r^{3+\varepsilon-1}\exp\left(-2\sqrt{\gamma_1}r\right)dr + \left(\varepsilon^2 + 2\varepsilon + 1\right)\int_0^{+\infty} r^{2+\varepsilon-1}\exp\left(-2\sqrt{\gamma_1}r\right)\right\} \\ T_2(n=1,\varepsilon,\gamma_1) &= \frac{\beta - \alpha V_0}{2}\int_0^{+\infty} r^{1+\varepsilon-1}\exp\left(-2\sqrt{\gamma_1}r\right)\left(4\gamma_1 r^2 - 2\sqrt{\gamma_1}(\varepsilon+2)r + \varepsilon^2 + 2\varepsilon + 1\right)dr \end{aligned} \tag{31}$$

Evaluating the integral in Eqn. (31), applies the special integration, which given by Eqn. (29), we obtain the results:

$$T_1(n=1,\varepsilon,\gamma_1)=\left(\alpha^2V_0+\frac{1}{2}\mu\omega^2\right)T_3(n=1,\varepsilon,\gamma_1)$$
$$=\left(\alpha^2V_0+\frac{1}{2}\mu\omega^2\right)\left\{(4\gamma_1)^{-(6+\varepsilon)/2}\Gamma(4+\varepsilon)-2\sqrt{\gamma_1}(\varepsilon+2)(4\gamma_1)^{-(3+\varepsilon)/2}\Gamma(3+\varepsilon)+(\varepsilon^2+2\varepsilon+1)(4\gamma_1)^{-(2+\varepsilon)/2}\Gamma(2+\varepsilon)\right\} \quad (32)$$
$$T_2(n=1,\varepsilon,\gamma_1)=\frac{\beta-\alpha V_0}{2}\left\{(4\gamma_1)^{-(5+\varepsilon)/2}\Gamma(3+\varepsilon)-2\sqrt{\gamma_1}(\varepsilon+2)(4\gamma_1)^{-(2+\varepsilon)/2}\Gamma(2+\varepsilon)+(\varepsilon^2+2\varepsilon+1)(4\gamma_1)^{-(1+\varepsilon)/2}\Gamma(1+\varepsilon)\right\}$$

This allows us to obtain the exact modifications $E_{\text{u-}mgo}(n=1,\varepsilon,\gamma_1)$ and $E_{\text{d-}mgo}(n=1,\varepsilon,\gamma_1)$ of the first excited state, and in the same way we find the exact modifications $E_{\text{u-}mgo}(n,\varepsilon,\gamma,l)$ and $E_{\text{d-}mgo}(n,\varepsilon,\gamma,l)$ for n^{th} excited states of Hydrogenic atoms with MQMGHO potential in global quantum group symmetry (NC: 3D-RSP).

3.4 The exact modified magnetic spectrum for MQMGHO potential in global (NC: 3D- RSP) symmetries

Further to previously obtained important results we now consider another physically meaningful phenomena produced by the effect of MQMGHO potential related to the influence of an external uniform magnetic field $\vec{B}$, to avoid the repetition in the theoretical calculations, it's sufficient to apply the following replacements:

$$\begin{cases}\vec{\Theta}\to\chi\vec{B}\\ \vec{\bar{\theta}}\to\bar{\sigma}\vec{B}\end{cases}\Rightarrow\left\{-\left(\alpha^2V_0+\frac{1}{2}\mu\omega^2+\frac{\beta-\alpha V_0}{2r}\right)\Theta+\frac{\bar{\theta}}{2\mu}\right\}$$

$$\text{replace\_by}\left\{-\left(\alpha^2V_0+\frac{1}{2}\mu\omega^2+\frac{\beta-\alpha V_0}{2r}\right)\chi+\frac{\bar{\sigma}}{2\mu}\right\}\vec{B}\vec{L} \quad (33)$$

Here χ and $\bar{\sigma}$ are two infinitesimal real proportional constants, and we choose the arbitrary external magnetic field $\vec{B}$ parallel to the (Oz) axis, which allow us to introduce the new modified magnetic Hamiltonian H_{m-mg} in (NC: 3D-RSP) symmetries as

$$H_{m-mgo}=\left(-\left(\alpha^2V_0+\frac{1}{2}\mu\omega^2+\frac{\beta-\alpha V_0}{2r}\right)\chi+\frac{\bar{\sigma}}{2\mu}\right)\left\{\vec{B}\vec{J}-\aleph_z\right\} \quad (34)$$

Here, $\aleph_z\equiv-\vec{S}\vec{B}$ denotes the Zeeman effect, whereas $\aleph_{\text{mod}-z}\equiv\vec{B}\vec{J}-\aleph_z$ is the new Zeeman effect. To obtain the exact NC magnetic modifications of energy for ground state, first excited state and n^{th} excited states of Hydrogenic atoms $E_{mag-mgo}(n=0,m=0,\varepsilon,\gamma_1)$, $E_{mag-mgo}(n=1,m=0,\pm1,\varepsilon,\gamma_1)$ and $E_{mag-mgo}(n,m,\varepsilon,\gamma_1)$ we just replace k_+ and Θ in the Eq. (27) by the following parameters: m and χ, respectively, as

$$E_{mag-mgo}(n=0,m=0,\varepsilon,\gamma_1)=0$$
$$E_{mag-mgo}(n=1,m=0,\pm1,\varepsilon,\gamma_1)=-\gamma N^2\begin{Bmatrix}\chi T_1(n=1,\varepsilon,\gamma_1)+\chi T_2(n=1,\varepsilon,\gamma_1)\\ -\dfrac{\bar{\sigma}}{2\mu}T_3(n,\varepsilon,\gamma_1)\end{Bmatrix}Bm$$
$$E_{mag-mgo}(n,m,\varepsilon,\gamma_1)=-\gamma N^2\left\{\chi T_1(n,\varepsilon,\gamma_1)+\chi T_2(n,\varepsilon,\gamma_1)-\frac{\bar{\sigma}}{2\mu}T_3(n,\varepsilon,\gamma_1)\right\}Bm$$
(35)

We have $-l\leq m\leq+l$, which allows us to fix ($2l+1$) values for discreet numbers m.

3. Results

In this section, we discuss several results obtained in the previous section, our goal from this work is on focusing around the modified eigen-energies ($E_{\text{nc-u}mgo}(n=0,j,l,s,\varepsilon,\gamma_1)$, $E_{\text{nc-d}mgo}(n=0,j,l,s,\varepsilon,\gamma_1)$), ($E_{\text{nc-u}mgo}(n=1,j,l,s,\varepsilon,\gamma_1)$, $E_{\text{nc-d}mgo}(n=1,j,l,s,\varepsilon,\gamma_1)$) and ($E_{\text{nc-u}mgo}(n,j,l,s,\varepsilon,\gamma_1)$, $E_{\text{nc-d}mgo}(n,j,l,s,\varepsilon,\gamma_1)$) of a Hydrogenic atoms with spin $\vec{S}=1/2$ for MSE with MQMGHO potential are obtained in this paper on based on our original results presented by Eqns. (27), (31), (32) and (35), in addition to the ordinary energy E_n for QMGHO potential, which is presented by Eqn. (10):

$$E_{\text{nc-u}mgo}(n=0,j,l,s,\varepsilon,\gamma_1)=E_0$$
$$E_{\text{nc-d}mgo}(n=0,j,l,s,\varepsilon,\gamma_1)=E_0-\frac{\gamma N^2}{2}\left\{\Theta T_1(n=0,\varepsilon,\gamma_1)+\Theta T_2(n=0,\varepsilon,\gamma_1)-\frac{\bar{\theta}}{2\mu}T_3(n=0,\varepsilon,\gamma_1)\right\} \quad (36)$$

$$E_{\text{nc-u}mgo}(n=1,j,l,s,\varepsilon,\gamma_1)=E_1-\gamma N^2\left\{(k_+\Theta+\chi Bm)(T_1(n=1,\varepsilon,\gamma_1)+\Theta T_2(n=1,\varepsilon,\gamma_1))-\left(\frac{\bar{\theta}}{2\mu}k_++\frac{\bar{\sigma}}{2\mu}Bm\right)T_3(n=1,\varepsilon,\gamma_1)\right\}$$
$$E_{\text{nc-d}mgo}(n=1,j,l,s,\varepsilon,\gamma_1)=E_1-\gamma N^2\left\{(k_-\Theta+\chi Bm)(T_1(n=1,\varepsilon,\gamma_1)+\Theta T_2(n=1,\varepsilon,\gamma_1))-\left(\frac{\bar{\theta}}{2\mu}k_++\frac{\bar{\sigma}}{2\mu}Bm\right)T_3(n=1,\varepsilon,\gamma_1)\right\} \tag{37}$$

$$E_{\text{nc-u}mgo}(n,j,l,s,\varepsilon,\gamma_1)=E_n-\gamma N^2\left\{(k_+\Theta+\chi Bm)(T_1(n,\varepsilon,\gamma_1)+\Theta T_2(n,\varepsilon,\gamma_1))-\left(\frac{\bar{\theta}}{2\mu}k_++\frac{\bar{\sigma}}{2\mu}Bm\right)T_3(n,\varepsilon,\gamma_1)\right\}$$
$$E_{\text{nc-d}mgo}(n,j,l,s,\varepsilon,\gamma_1)=E_n-\gamma N^2\left\{(k_-\Theta+\chi Bm)(T_1(n,\varepsilon,\gamma_1)+\Theta T_2(n,\varepsilon,\gamma_1))-\left(\frac{\bar{\theta}}{2\mu}k_++\frac{\bar{\sigma}}{2\mu}Bm\right)T_3(n,\varepsilon,\gamma_1)\right\} \tag{38}$$

Where

$$E_0=V_0+\frac{1}{2\mu}\left\{1/4+\frac{\mu(\beta-\alpha V_0)}{\sqrt{2\mu(\alpha^2V_0+\mu\omega^2/2)}}+\frac{\mu(\beta-\alpha V_0)^2}{2[2\mu(\alpha^2V_0+\mu\omega^2/2)]}\right\}$$
$$E_1=V_0+\frac{1}{2\mu}\left\{9/4+\frac{3\mu(\beta-\alpha V_0)}{\sqrt{2\mu(\alpha^2V_0+\mu\omega^2/2)}}+\frac{\mu(\beta-\alpha V_0)^2}{2[2\mu(\alpha^2V_0+\mu\omega^2/2)]}\right\} \tag{39}$$
$$E_n=V_0+\frac{1}{2\mu}\left\{(n+1/2)\left[(n+1/2)+\frac{2\mu(\beta-\alpha V_0)}{\sqrt{2\mu(\alpha^2V_0+\mu\omega^2/2)}}\right]+\frac{\mu(\beta-\alpha V_0)^2}{2[2\mu(\alpha^2V_0+\mu\omega^2/2)]}\right\}$$

This is the main goal of this work. It's clear, that the obtained eigenvalues of energies are reals and then the non-commutative diagonal Hamiltonian H_{nc-mgo} is Hermitian, furthermore it's possible to write the following three elements $(H_{nc-mgo})_{11}$, $(H_{nc-mgo})_{22}$ and $(H_{nc-mgo})_{33}$ as follows:

$$(H_{nc-mgo})_{11}=-\frac{\Delta_{nc}}{2\mu}+H_{\text{int}-umgo}$$
$$(H_{nc-mgo})_{22}=-\frac{\Delta_{nc}}{2\mu}+H_{\text{int}-dmgo} \tag{40}$$
$$(H_{nc-mgo})_{33}=H_{mgo}$$

$$\frac{\Delta_{nc}}{2\mu}=\frac{\Delta-\bar{\theta}\vec{L}-\bar{\sigma}\vec{L}}{2\mu}$$
$$H_{\text{int}-umgo}=V_{mgo}(r)-\gamma(k_+\Theta+\chi\aleph_{\text{mod}-z})f(r) \tag{41}$$
$$H_{\text{int}-dmgo}=V_{mgo}(r)-\gamma(k_-\Theta+\chi\aleph_{\text{mod}-z})f(r)$$

Where,

Thus, the ordinary kinetic term for QMGHO potential $-\frac{\Delta}{2\mu}$ and ordinary interaction $V_0+(\beta-\alpha V_0)r+\left(\alpha^2V_0+\frac{1}{2}\mu\omega^2\right)r^2$ are replaced by a new modified form of a kinetic term $\frac{\Delta_{nc}}{2\mu}$ (which generate the dynamic of the physical system) and two modified interactions to the new form ($H_{\text{int}-umgo}$ and $H_{\text{int}-dmgo}$). On the other hand, it is evident to consider the quantum number m takes ($2l+1$) values and we have also two values for $j=l\pm\frac{1}{2}$, thus every state in usually three dimensional space of energy for MQMGHO potential will be $2(2l+1)$ sub-states. To obtain the total complete degeneracy of energy level of the MQMGHO potential in NC three-dimension spaces-phases, we need to sum for all allowed values of l. Total degeneracy is thus,

$$2\sum_{l=0}^{n-1}(2l+1)\equiv 2n^2 \tag{42}$$

Note that the obtained new energy eigenvalues ($E_{\text{nc-u}mgo}(n,j,l,s,\varepsilon,\gamma_1)$, $E_{\text{nc-d}mgo}(n,j,l,s,\varepsilon,\gamma_1)$) now depend to new discrete atomic quantum numbers (n, j, l, s) and m in addition to the parameter α of the potential. It is pertinent to note that when the atoms have $\vec{S} \neq \vec{1/2}$, the total operator can be obtains from the interval $|l-s| \leq j \leq |l+s|$, which allow us to obtaining the eigenvalues of the operator $\left(\overleftrightarrow{J}^2 - \overleftrightarrow{L}^2 - \overleftrightarrow{S}^2\right)$ as $k(j,l,s) \equiv j(j+1) + l(l+1) - s(s+1)$ and then the nonrelativistic energy spectrum $E_{\text{nc-}mgo}(n, j, l, s, \varepsilon, \gamma_1)$ reads:

$$E_{\text{nc-}mgo}(n,j,l,s,\varepsilon,\gamma_1) = E_n - \gamma N^2\left\{\left(k(,l,s)\Theta + \chi Bm\right)\left(T_1(n,\varepsilon,\gamma_1) + \Theta T_2(n,\varepsilon,\gamma_1)\right) - \left(\frac{\overline{\theta}}{2\mu}k_+ + \frac{\overline{\sigma}}{2\mu}Bm\right)T_3(n,\varepsilon,\gamma_1)\right\} \quad (43)$$

If $\beta = V_0$ in Eq. (19), the new Hamiltonian operator $H_{\text{nc-}mgo}(\hat{r})$ turns back into the modified harmonic oscillator Hamiltonian operator in (NC: 3D- RSP) symmetries and the energy equation (43) yields the energy eigenvalues for the modified harmonic oscillator potential. If we consider $(\Theta, \chi) \to (0,0)$, we recover the results of commutative space of refs [7-8] obtained for the quantum mechanical gravitational potential plus the harmonic oscillator potential

5. Conclusion

In this paper, we have studied the bound state solution of the modified Schrödinger equation for modified quantum mechanical gravitational potential plus the modified harmonic oscillator potential via the generalized Bopp's shift method and standard perturbation theory in (NC: 3D-RSP) symmetries. We resume the main obtained results:

- We have seen that the modified Hamiltonian operator containing a perturbed Columbic potential, this means physically the global Hamiltonian can be described Hydrogenic atoms interacted with strong potential composed with quantum mechanical gravitational potential plus the harmonic oscillator potential and an auxiliary part.
- Ordinary interaction ($V_0 + (\beta - \alpha V_0)r + \left(\alpha^2 V_0 + \frac{1}{2}\mu\omega^2\right)r^2$) were replaced by new modified interactions $H_{\text{int}-umgo}$ and $H_{\text{int}-dmgo}$ for Hydrogenic atoms,
- The ordinary kinetic term ($-\frac{\Delta}{2\mu}$) modified to the new form $\frac{\Delta_{nc}}{2\mu} = \frac{\Delta - \overline{\theta}\vec{L} - \overline{\sigma}\vec{L}}{2\mu}$ for MQGHO potential,
- The modified eigenenergies ($E_{\text{nc-u}mgo}(n=0,j,l,s,\varepsilon,\gamma_1)$, $E_{\text{nc-d}mgo}(n=0,j,l,s,\varepsilon,\gamma_1)$), ($E_{\text{nc-u}mgo}(n=1,j,l,s,\varepsilon,\gamma_1)$, $E_{\text{nc-d}mgo}(n=1,j,l,s,\varepsilon,\gamma_1)$) and ($E_{\text{nc-u}mgo}(n,j,l,s,\varepsilon,\gamma_1)$, $E_{\text{nc-d}mgo}(n,j,l,s,\varepsilon,\gamma_1)$) of a Hydrogenic atoms with spin $\vec{S} = \vec{1/2}$ for MSE with MQMGHO potential are obtained.

Through this research, we can conclude the generalized Bopp's shift method can be applied to the investigation of other physical systems in NCQM symmetries.

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