

New Relativistic Bound States for Modified Pseudoharmonic Potential of Dirac Equation with Spin and Pseudo-Spin Symmetry in One-electron Atoms

Abdelmadjid Maireche*

*Laboratory of Physics and Material Chemistry, Physics Department, Sciences Faculty,
University of M'sila-M'sila, Algeria*

In this paper, we present further results of our investigation for the exact solvability of relativistic quantum systems with modified pseudo-harmonic (M.P.H.) potential for spin-1/2 particles by of means Bopp's shift method instead of solving deformed Dirac equation with star product, in the framework of noncommutative 3-dimensional real space (NC: 3D-RS) symmetries. The exact corrections for excited n^{th} states are found straightforwardly for interactions in one-electron atoms by means of the standard perturbation theory. Furthermore, the obtained corrections of energies are depended on two infinitesimal parameter Θ and χ which are induced by position-position noncommutativity, in addition to the discrete atomic quantum numbers: $j = \tilde{l}(l) \pm 1/2, s = \pm 1/2, \tilde{l}(l)$ and $\tilde{m}(m)$ (angular momentum quantum numbers). We have also shown that the usual states in ordinary three dimensional spaces are canceled and replaced by new degenerated $2(2\tilde{l} + 1)$ and $2(2l + 1)$ sub-states under the pseudo spin symmetry and spin symmetry conditions respectively in the new quantum symmetries of (NC: 3D-RS).

1. Introduction

Recently, the exact analytical solutions of Schrödinger (for fermions with spin 1/2) Klein-Gordon equation (for bosons with spin zero) and Dirac (fermionic particle and anti-particle with spin 1/2) equations for some physical central and non-central potentials were shown to be essential because the knowledge of wave functions and energy contains all possible important information of the physical properties of quantum system for both nonrelativistic and relativistic quantum mechanics [1-41]. The quantum algebraic structure based on the ordinary canonical commutations relations (CCRs) in both Schrödinger and Heisenberg (operators are depended on time) pictures, respectively, (in $c = \hbar = 1$ units) as

$$\begin{cases} [x_i, p_j] = i\delta_{ij} & \text{and} & [x_i, x_j] = [p_i, p_j] = 0 \\ [x_i(t), p_j(t)] = i\delta_{ij} & \text{and} & [x_i(t), x_j(t)] = [p_i(t), p_j(t)] = 0 \end{cases} \quad (1)$$

Where the two operators $(x_i(t), p_i(t))$ in Heisenberg picture are related to the corresponding two operators (x_i, p_i) in Schrödinger picture from the two projections relations, respectively [61] as

$$\begin{aligned} x_i(t) &= \exp(i\hat{H}_{ph}(t-t_0))x_i \exp(-i\hat{H}_{ph}(t-t_0)) \\ p_i(t) &= \exp(i\hat{H}_{ph}(t-t_0))p_i \exp(-i\hat{H}_{ph}(t-t_0)) \end{aligned} \quad (2)$$

Here \hat{H}_{ph} denote to the ordinary quantum Hamiltonian operator for pseudo-harmonic potential. H. Snyder was the first to introduce the noncommutativity idea for almost seventy years ago [42] and very recently the non-commutative geometry played an important role in modern physics and has sustained great interest [43-76]. The new quantum structure of noncommutative space based on the following noncommutative canonical commutations relations (NCCRs) in both Schrödinger and Heisenberg pictures, respectively, as follows [43-73]

$$\begin{cases} [\hat{x}_i^*, \hat{p}_j] = i\delta_{ij}, [\hat{x}_i^*, \hat{x}_j] = i\theta_{ij} & \text{and} & [\hat{p}_i^*, \hat{p}_j] = 0 \end{cases} \quad (3)$$

and

$$[\hat{x}_i(t)^*, \hat{p}_j(t)] = i\delta_{ij}, [\hat{x}_i(t)^*, \hat{x}_j(t)] = i\theta_{ij} \text{ and } [\hat{p}_i(t)^*, \hat{p}_j(t)] = 0 \quad (4)$$

Where the two new operators $(\hat{x}_i(t), \hat{p}_i(t))$ in Heisenberg picture are related to the corresponding two new operators (\hat{x}_i, \hat{p}_i) in Schrödinger picture

*abmaireche@gmail.com

from the two projections relations, respectively [61] as

$$\begin{aligned}\hat{x}_i(t) &= \exp(i\hat{H}_{nc-ph}(t-t_0)) * \hat{x}_i * \exp(-i\hat{H}_{nc-ph}(t-t_0)) \\ \hat{p}_i(t) &= \exp(i\hat{H}_{nc-ph}(t-t_0)) * \hat{p}_i * \exp(-i\hat{H}_{nc-ph}(t-t_0))\end{aligned}\quad (5)$$

Here, \hat{H}_{nc-ph} denote to the new quantum Hamiltonian operator in the symmetries of (NC: 3D-RS). $\theta^{\mu\nu}$ has very small parameters (compared to the energy) that are elements of antisymmetric real matrix and (*) denotes the new star product, which is generalized between two arbitrary functions $f(x) \rightarrow \hat{f}(\hat{x})$ and $g(x) \rightarrow \hat{g}(\hat{x})$ to $\hat{f}(\hat{x})\hat{g}(\hat{x}) \equiv (f * g)(x)$ instead of the usual product $(fg)(x)$ in ordinary three dimensional spaces [43-57]

$$\begin{aligned}\hat{f}(\hat{x})\hat{g}(\hat{x}) &\equiv (f * g)(x) \equiv \exp\left(\frac{i}{2}\theta^{\mu\nu}\partial_\mu^x\partial_\nu^x\right)(fg)(x, p) \\ &\equiv (fg - \frac{i}{2}\theta^{\mu\nu}\partial_\mu^x\partial_\nu^x fg) \Big|_{(x^\mu = x^{\nu\nu})} + O(\theta^2)\end{aligned}\quad (6)$$

Where $\hat{f}(\hat{x})$ and $\hat{g}(\hat{x})$ are the new functions in (NC: 3D-RS), $\partial_\mu^x f(x)$ denotes $\frac{\partial f(x)}{\partial x^\mu}$ and the following term $(-\frac{i}{2}\theta^{\mu\nu}\partial_\mu^x\partial_\nu^x f(x)\partial_\nu^x g(x))$ is induced by (space-space) noncommutativity properties and $O(\theta^2)$ stands for the second and higher order terms of θ . The Bopp's shift method can be used, instead of solving any quantum systems by using directly star product procedure [43-65]

$$[\hat{x}_i, \hat{x}_j] = i\theta_{ij} \text{ and } [\hat{p}_i, \hat{p}_j] = 0 \quad (7)$$

The three-generalized coordinates $(\hat{x} = \hat{x}_1, \hat{y} = \hat{x}_2, \hat{z} = \hat{x}_3)$ in the noncommutative space are depended on corresponding three-usual generalized positions (x, y, z) and momentum coordinates (p_x, p_y, p_z) by the following relations, as follows [61]

$$\begin{aligned}\hat{x} &= x - \frac{\theta_{12}}{2} p_y - \frac{\theta_{13}}{2} p_z, & \hat{y} &= y - \frac{\theta_{21}}{2} p_x - \frac{\theta_{23}}{2} p_z \\ \hat{z} &= z - \frac{\theta_{31}}{2} p_x - \frac{\theta_{32}}{2} p_y\end{aligned}\quad (8)$$

The non-vanishing commutators in (NC-3D: RS) can be determined as follows

$$\begin{aligned}[\hat{x}, \hat{p}_x] &= [\hat{y}, \hat{p}_y] = [\hat{z}, \hat{p}_z] = i, \\ [\hat{x}, \hat{y}] &= i\theta_{12}, [\hat{x}, \hat{z}] = i\theta_{13}, [\hat{y}, \hat{z}] = i\theta_{23}\end{aligned}\quad (9)$$

which allow us to getting the operator \hat{r}^2 on noncommutative three dimensional spaces as follows [47,61,63,65]:

$$\hat{r}^2 = r^2 - \vec{\mathbf{L}}\vec{\Theta} \quad (10)$$

Where the coupling $\mathbf{L}\Theta \equiv L_x\Theta_{12} + L_y\Theta_{23} + L_z\Theta_{13}$

and $\left(\Theta_{ij} = \frac{\theta_{ij}}{2}\right)$, in particular, the pseudo-

harmonic potentials have the general features of the true interaction energy, inter atomic and dynamical properties in solid-state physics and play an important role in the history of molecular structures and interactions; this potential is considered as an intermediate between harmonic oscillator and Morse-type potentials which are more realistic anharmonic potentials. Furthermore, the pseudo-harmonic potential is extensively used to describe the bound state of the interaction systems, and has been applied for both classical and modern physics [40]. This work is aimed at obtaining an analytic expression for the eigenenergies of a modified pseudoharmonic potential (M.P.H.), the potential in (NC: 3D-RS) symmetries using the generalization of Bopp's shift method to discover the new symmetries and a possibility to obtain another application to this potential in different fields. This work is based essentially on our previous works [43-65] and it was considered in our work [65] in the case of nonrelativistic case. The organization scheme is given as follows. In the next section, we briefly review the Dirac equation with pseudoharmonic potential on based to Ref. [41]. Sec. 3 is devoted to studying the three deformed Dirac equation by applying Bopp's shift method. In the fourth section by applying standard perturbation theory we find the quantum spectrum of the n^{th} excited states in (NC-3D: RS) for relativistic spin-orbital interaction. In the next

section, we derive the magnetic spectrum for the studied potential. In the Sec. 6, we resume the global spectrum and corresponding noncommutative Hamiltonian for (M.P.H.) potential. Finally, the important results and the conclusions are discussed in last section.

2. Review of the Dirac Equation for Pseudo-harmonic Potential

In this section, we shall review the eigenfunctions and eigenvalues for spherically symmetric for the pseudoharmonic potential $V(r)$ for the spin symmetric case and the pseudo-spin spin-symmetry [41]:

$$V(r) = D_0 \left(\frac{r}{r_0} - \frac{r_0}{r} \right)^2 \equiv ar^2 + \frac{b}{r^2} + c \quad (11)$$

Where, D_0 and r_0 are constant related to the dissociation energy of a molecule and an equilibrium distance, respectively, while $a = D_0 r_0^{-2}$, $b = D_0 r_0^{+2}$ and $c = -2D_0$. The Dirac equation in the presence of above interaction is given by [41]

$$(\alpha p + \beta(M + S(r)))\Psi(r, \theta, \varphi) = (E - V(r))\Psi(r, \theta, \varphi) \quad (12)$$

here M , E and $(\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \beta = \begin{pmatrix} I_{2 \times 2} & 0 \\ 0 & I_{2 \times 2} \end{pmatrix})$ are the fermions' mass, the relativistic energy and the usual Dirac matrices are: $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and are 2×2 three Pauli matrices, thus the ordinary Hamiltonian operator \hat{H}_{0ph} can be expressed as

$$\hat{H}_{0ph} = (\alpha p + \beta(M + S(r))) + V(r) \quad (13)$$

The spinor $\Psi(r, \theta, \varphi)$ can be written as [41]

$$\Psi_{nk}(r, \theta, \varphi) = \begin{pmatrix} f_{nk} \begin{pmatrix} \uparrow \\ r \end{pmatrix} \\ g_{nk} \begin{pmatrix} \uparrow \\ r \end{pmatrix} \end{pmatrix} = \frac{1}{r} \begin{pmatrix} F_{nk}(r) Y_{jm}^l(\theta, \varphi) \\ i G_{nk}(r) Y_{jm}^{\tilde{l}}(\theta, \varphi) \end{pmatrix} \quad (14)$$

Where, $F_{nk}(r)$ and $G_{nk}(r)$ are the upper and

lower components of the Dirac spinors, $Y_{jm}^l(\theta, \varphi)$ and $Y_{jm}^{\tilde{l}}(\theta, \varphi)$ are the spin and pseudo-spin spherical harmonics while k (\tilde{k}) is related to the total angular momentum quantum numbers for spin symmetry l and p-spin symmetry \tilde{l} as [38,39,41]

$$k = \begin{cases} -(l+1) & \text{if } -(j+1/2), (s_{1/2}, p_{3/2}, etc), \\ j = l + \frac{1}{2}, \text{aligned spin}(k>0) \\ +l & \text{if } j = l + \frac{1}{2}, (p_{1/2}, d_{3/2}, etc), \\ j = l - \frac{1}{2}, \text{unaligned spin}(k>0) \end{cases} \quad (15)$$

and

$$\tilde{k} = \begin{cases} -\tilde{l} & \text{if } -(j+1/2), (s_{1/2}, p_{3/2}, etc), \\ j = \tilde{l} - \frac{1}{2}, \text{aligned spin}(k>0) \\ +(\tilde{l} + 1) & \text{if } j = \tilde{l} + \frac{1}{2}, (p_{1/2}, d_{3/2}, etc), \\ j = \tilde{l} + \frac{1}{2}, \text{unaligned spin}(k>0) \end{cases} \quad (16)$$

The radial functions ($F_{nk}(r)$, $G_{nk}(r)$) are obtained by solving the following differential equations [41]

$$\left[\frac{d^2}{dr^2} - \frac{k(k+1)}{r^2} - [M + E_{nk} - \Delta(r)][M - E_{nk} + \Sigma(r)] + \frac{d\Delta(r)}{dr} \left(\frac{d}{dr} + \frac{k}{r} \right) \right] F_{nk}(r) = 0 \quad (17)$$

$$\left[\frac{d^2}{dr^2} - \frac{k(k-1)}{r^2} - [M + E_{nk} - \Delta(r)][M - E_{nk} + \Sigma(r)] + \frac{d\Delta(r)}{dr} \left(\frac{d}{dr} - \frac{k}{r} \right) \right] G_{nk}(r) = 0 \quad (18)$$

The bound state solutions of the pseudo-harmonic potential for the spin symmetric case obtained in the exact spin symmetry $\frac{d\Delta(r)}{dr} = 0$ and then the energy eigenvalues depend on n and l . According to the Laplace transform approach (LTA) and asymptotic interaction method, which was applied in Refs. [41], the upper component $F_{nk}(r)$ of the Dirac spinors are

$$F_{nk}(r) = Nr^{\nu+1} e^{-\frac{\mu r^2}{2}} {}_1F_1\left(-n, \nu + \frac{3}{2}, r^2\right) \quad (19)$$

Where, N and ${}_1F_1(-n, 2\gamma + 1, 2\epsilon r)$ are the normalization constant and the confluent hypergeometric functions, the relativistic positive energy eigenvalues with the pseudo-harmonic potential under the spin-symmetry condition are obtained as [41]

$$\frac{r_0}{\sqrt{D_0}} (2D_0 + E - M) \sqrt{M + E - C} - \sqrt{(2k+1)^2 + 4D_0^2 r_0^2 (M + E - C)} = 4(n+1/2) \quad (20)$$

$n = 0, 1, 2, \dots$. In the exact pseudo-spin symmetric case, the lower $G_{nk}(r)$ component of the Dirac spinors [40]

$$G_{nk}(r) = \tilde{N} r^{\nu+1} e^{-\frac{\mu r^2}{2}} {}_1F_1\left(-n, \nu + \frac{3}{2}, r^2\right) \quad (21)$$

With $\nu(\nu+1) = k(k+1) + (M + E - C)D_0 r_0^2$ and $\mu^2 = \frac{D_0}{r_0^2} (M + E - C)$ while \tilde{N} denote to the normalization constant and the relativistic negative energy eigenvalues with the pseudoharmonic potential under the pseudo-spin spin-symmetry condition is obtained as [41]

$$\frac{r_0}{\sqrt{D_0}} (2D_0 + E - M) \sqrt{E - M - C} - \sqrt{(2k+1)^2 + 4D_0^2 r_0^2 (M - E + C)} = 4(n+1/2) \quad (22)$$

The generalized Laguerre polynomials $L_n^{(p)}(x)$ can be expressed as a function of the confluent hypergeometric functions as

$$L_n^{(p)}(x) = \frac{\Gamma(n+p+1)}{n! \Gamma(p+1)} {}_1F_1(-n, p+1; x) \quad (23)$$

Which allow us to rewrite the upper component $F_{nk}(r)$ and the lower $G_{nk}(r)$ component of the Dirac spinors for the spin symmetric case and the pseudo-spin spin-symmetry, respectively, as

$$F_{nk}(r) = Nn! \frac{\Gamma\left(\nu + \frac{3}{2}\right)}{\Gamma\left(n + \nu + \frac{3}{2}\right)} r^{\nu+1} e^{-\frac{\mu r^2}{2}} L_n^{\left(\nu + \frac{1}{2}\right)}(r^2) \quad (24)$$

and

$$G_{nk}(r) = \tilde{N}n! \frac{\Gamma\left(\nu + \frac{3}{2}\right)}{\Gamma\left(n + \nu + \frac{3}{2}\right)} r^{\nu+1} e^{-\frac{\mu r^2}{2}} L_n^{\left(\nu + \frac{1}{2}\right)}(r^2) \quad (25)$$

3. Noncommutative relativistic Hamiltonian operator for (M.P.H.) potential

3.1. Formalism of Bopp's shift method

Now, we shall review some fundamental principles of the quantum noncommutative Dirac equation, which resumed in the following steps for modified pseudo-harmonic potential $V(\hat{r})$ [47,61,63,65] as

- Ordinary Hamiltonian $\hat{H}(p_i, x_i)$ replaced by noncommutative Hamiltonian $\hat{H}(\hat{p}_i, \hat{x}_i)$,
- Ordinary spinor $\Psi(\vec{r})$ replaced by new spinor $\hat{\Psi}(\vec{r})$,
- Ordinary energy E replaced by new energy E_{nc-ph} , and ordinary product replaced by new star product $*$.

These previous steps allow us to write the new noncommutative Dirac equation for modified pseudo-harmonic potential as follows

$$\hat{H}(\hat{p}_i, \hat{x}_i) * \hat{\Psi}(\vec{r}) = E_{nc-ph} \hat{\Psi}(\vec{r}) \quad (26)$$

It is worth emphasizing that the Bopp's shift method permutes to reduce the above equation to simplest their form

$$H_{nc-ph}(\hat{p}_i, \hat{x}_i) \psi(\vec{r}) = E_{nc-ph} \psi(\vec{r}) \quad (27)$$

Where, $\psi(\vec{r})$ is a solution of the Dirac equation and the new operator of Hamiltonian $H_{nc-ph}(\hat{p}_i, \hat{x}_i)$ can be expressed in three general varieties: both noncommutative space and noncommutative phase (NC-3D: RSP), only noncommutative space (NC-3D: RS) and only noncommutative phase (NC: 3D-RP), respectively, [65] as

$$H_{nc-ph}(\hat{p}_i, \hat{x}_i) \equiv H\left(\hat{p}_i = p_i - \frac{1}{2}\theta_{ij}x_j; \hat{x}_i = x_i - \frac{1}{2}\theta_{ij}p_j\right) \text{ for NG-3DRS1} \dots (28)$$

$$H_{nc-ph}(\hat{p}_i, \hat{x}_i) \equiv H\left(\hat{p}_i = p_i; \hat{x}_i = x_i - \frac{1}{2}\theta_{ij}p_j\right) \text{ for NC-3D:RS} (29)$$

$$H_{nc-ph}(\hat{p}_i, \hat{x}_i) \equiv H\left(\hat{p}_i = p_i - \frac{1}{2}\theta_{ij}x_j; \hat{x}_i = x_i\right) \text{ for NC-3D:RP} (30)$$

In a recent work, we were interested in the above second variety and then the new modified Hamiltonian $H_{nc-ph}(\hat{p}_i, \hat{x}_i)$ defined as a function of $\hat{x}_i = x_i - \frac{1}{2}\theta_{ij}p_j$ and $\hat{p}_i = p_i$:

$$H_{nc-ph}(\hat{p}_i, \hat{x}_i) = \alpha\hat{P} + \beta(M + S(\hat{r})) + V(\hat{r}) (31)$$

Where the modified pseudo-harmonic potential $V(\hat{r})$ is given by

$$V(\hat{r}) = \frac{a}{\hat{r}^2} - \frac{b}{\hat{r}} + c (32)$$

The Dirac equation in the presence of above interaction $V(\hat{r})$ can be rewritten according Boopp's shift method as follows

$$(\alpha P + \beta(M + S(\hat{r})))\Psi(r, \theta, \varphi) = (E - V(\hat{r}))\Psi(r, \theta, \varphi) (33)$$

The radial functions ($F_{nk}(r), G_{nk}(r)$) are obtained, in the absence of tensor interaction, by solving two equations

$$\left[\frac{d}{dr} + \frac{k}{r}\right]F_{nk}(r) = [M + E_{nc-ph} - \Delta(\hat{r})]G_{nk}(r) (34)$$

$$\left[\frac{d}{dr} + \frac{k}{r}\right]G_{nk}(r) = [M - E_{nc-ph} + \Sigma(\hat{r})]F_{nk}(r) (35)$$

with $\Delta(\hat{r}) = V(\hat{r}) - S(\hat{r})$ and $\Sigma(\hat{r}) = V(\hat{r}) + S(\hat{r})$, eliminating $F_{nk}(r)$ and $G_{nk}(r)$ from Eqns. (34) and (35), we can obtain the following two Schrödinger-like differential equations as follows with (NC-3D:RS) symmetries as

$$\left[\frac{d^2}{dr^2} - \frac{k(k+1)}{r^2} - (M + E_{nc-ph} - \Delta(\hat{r}))(M - E_{nc-ph} + \Sigma(\hat{r}))\right]F_{nk}(r) = 0 (36)$$

$$\left[\frac{d^2}{dr^2} - \frac{k(k-1)}{r^2} - (M + E_{nc-ph} - \Delta(\hat{r}))(M - E_{nc-ph} + \Sigma(\hat{r}))\right]G_{nk}(r) = 0 (37)$$

After straightforward calculations, one can obtain the two terms in (NC-3D:RS) spaces as follows

$$\begin{aligned} \frac{a}{\hat{r}^2} &= \frac{a}{r^2} + \frac{a}{r^4}\bar{\mathbf{L}}\bar{\Theta} + o(\theta^2) \\ -\frac{b}{\hat{r}} &= \frac{b}{r} - \frac{b}{2r^3}\bar{\mathbf{L}}\bar{\Theta} + o(\theta^2) \end{aligned} (38)$$

Which allows us to write the modified pseudo-harmonic potential $V(\hat{r})$ as follows

$$V(\hat{r}) = \frac{a}{r^2} - \frac{b}{r} + c + \begin{cases} \hat{V}_{1pert-ph}(r, \Theta, a, b) & \text{for thespinsymmetriccase} \\ \hat{V}_{2pert-ph}(r, \Theta, a, b) & \text{for thep-spinsymmetriccase} \end{cases} (39)$$

With

$$\begin{aligned} \hat{V}_{1pert-ph}(r, \Theta, a, b) &= \left(\frac{a}{r^4} - \frac{b}{2r^3}\right)\bar{\mathbf{L}}\bar{\Theta} \\ \hat{V}_{2pert-ph}(r, \Theta, a, b) &= \left(\frac{a}{r^4} - \frac{b}{2r^3}\right)\tilde{\mathbf{L}}\tilde{\Theta} \end{aligned} (40)$$

It is clearly that the star product inducing the non-commutativity is replaced by the usual product plus non local corrections $\hat{V}_{1pert-ph}(r, \Theta, a, b)$ and $\hat{V}_{2pert-ph}(r, \Theta, a, b)$ in the scalar potential $V(\hat{r})$.

This allows writing the modified Dirac equation in the non-commutative case as an equation similarly to the usual Dirac equation of the commutative type with a non local potential. Furthermore, using the unit step function (also known as the Heaviside step function or simply the theta function) we can rewrite the modified pseudoharmonic potential to the following form

$$V(\hat{r}) = \frac{a}{r^2} - \frac{b}{r} + c + \theta(E_{nc-ph}) \hat{V}_{1pert-ph}(r, \Theta, a, b) + \theta(-E_{nc-ph}) \hat{V}_{2pert-ph}(r, \Theta, a, b) \quad (41)$$

Where

$$\theta(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases} \quad (42)$$

We generalized the constraint for the pseudospin (p-spin) symmetry ($\Delta(r) = V(r)$ and $\Sigma(r) = C_{ps} = \text{constants}$ which presented in Refs. [38,39,40] into the new form $\Delta(\hat{r}) = V(\hat{r})$ and $\Sigma(\hat{r}) = \hat{C}_{ps} = \text{constants}$ in (NC-3D: RS) and inserting the potential $V(\hat{r})$ in Eqn. (41) into the two Schrödinger-like differential Eqns. (36) and (37), one obtains

$$\left[\frac{d^2}{dr^2} - \frac{k(k+1)}{r^2} - (M + E_{nc-ph})(M - E_{nc-ph} + \hat{C}_{ps}) - \left(\frac{a}{r^2} - \frac{b}{r} + c \right) (M - E_{nc-ph} + \hat{C}_{ps}) - \left(\frac{a}{r^4} - \frac{b}{2r^3} \right) \tilde{\mathbf{L}}\tilde{\Theta}(M - E_{nc-ph} + \hat{C}_{ps}) \right] F_{nk}(r) = 0 \quad (43)$$

$$\left[\frac{d^2}{dr^2} - \frac{k(k-1)}{r^2} - (M + E_{nc-ph})(M - E_{nc-ph} + \hat{C}_{ps}) - \left(\frac{a}{r^2} - \frac{b}{r} + c \right) (M - E_{nc-ph} + \hat{C}_{ps}) - \left(\frac{a}{r^4} - \frac{b}{2r^3} \right) \tilde{\mathbf{L}}\tilde{\Theta}(M - E_{nc-ph} + \hat{C}_{ps}) \right] G_{nk}(r) = 0 \quad (44)$$

and two similarly equations obtained by $\tilde{\mathbf{L}} \rightarrow \tilde{\tilde{\mathbf{L}}}$. It's clearly that, the additive new parts $\hat{V}_{1pert-ph}(r, \Theta, a, b)$ and $\hat{V}_{2pert-ph}(r, \Theta, a, b)$ are proportional with infinitesimal parameter Θ , thus, we can considered as a perturbations terms. Our aim is to derive the energy spectrum for a moving charged particle in the presence of a potential given by (41) analytically in a very simple way.

4. The exact relativistic spin-orbital Hamiltonian and corresponding spectrum for (M.P.H.) Potential in (NC: 3D- RS) symmetries for excited n^{th} states for one-electron atoms

4.1. The exact relativistic spin-orbital Hamiltonian for (M.P.H.) potential in (NC: 3D- RS) symmetries for one-electron atoms

The result in Eqn. (40) can be rewritten in a more accessible physical form, we replace both $\tilde{\mathbf{L}}\tilde{\Theta}$ and $\tilde{\tilde{\mathbf{L}}}\tilde{\Theta}$ by $\tilde{\tilde{S}}\tilde{\tilde{L}}$ and $\tilde{\tilde{S}}\tilde{\tilde{L}}$ and then the two perturbative

terms $\hat{V}_{1pert-ph}(r, \Theta, a, b)$ and $\hat{V}_{2pert-ph}(r, \Theta, a, b)$ for the spin symmetric case and the pseudo-spin spin-symmetry, respectively, can be rewritten to the equivalent new form for (M.P.H.) potential

$$\hat{V}_{1pert-ph}(r, \Theta, a, b) = \Theta \left(\frac{a}{r^4} - \frac{b}{2r^3} \right) \tilde{\tilde{S}}\tilde{\tilde{L}} \quad (45)$$

$$\hat{V}_{2pert-ph}(r, \Theta, a, b) = \Theta \left(\frac{a}{r^4} - \frac{b}{2r^3} \right) \tilde{\tilde{S}}\tilde{\tilde{L}}$$

Furthermore, the above perturbative terms $\hat{V}_{1pert-ph}(r, \Theta, a, b)$ and $\hat{V}_{2pert-ph}(r, \Theta, a, b)$ can be rewritten in the following new equivalent form for (M.P.H.) potential

$$\hat{V}_{1pert-ph}(r, \Theta, a, b) = \frac{1}{2} \Theta \left(\frac{a}{r^4} - \frac{b}{2r^3} \right) \left(\tilde{\tilde{J}}^2 - \tilde{\tilde{L}}^2 - \tilde{\tilde{S}}^2 \right)$$

$$\hat{V}_{2pert-ph}(r, \Theta, a, b) = \frac{1}{2} \Theta \left(\frac{a}{r^4} - \frac{b}{2r^3} \right) \left(\tilde{\tilde{J}}^2 - \tilde{\tilde{L}}^2 - \tilde{\tilde{S}}^2 \right) \quad (46)$$

To the best of our knowledge, we just replaced the coupling spin-orbital (p-spin-orbital) $\tilde{\tilde{S}}\tilde{\tilde{L}}$ and $\tilde{\tilde{S}}\tilde{\tilde{L}}$ by the two expressions: $\frac{1}{2} \left(\tilde{\tilde{J}}^2 - \tilde{\tilde{L}}^2 - \tilde{\tilde{S}}^2 \right)$ and $\frac{1}{2} \left(\tilde{\tilde{J}}^2 - \tilde{\tilde{L}}^2 - \tilde{\tilde{S}}^2 \right)$, respectively, in relativistic quantum mechanics. The set $(H_{nc-ph}(\hat{p}_i, \hat{x}_i), J^2, L^2, \tilde{\tilde{S}}^2 \text{ and } J_z)$ forms a complete of conserved physics quantities and the spin-orbit quantum number k (\tilde{k}) is related to the quantum numbers for spin symmetry l and p-spin symmetry \tilde{l} as follows [36,37]

$$\tilde{k} = \begin{cases} \tilde{k}_1 \equiv -\tilde{l} & \text{if } -(j+1/2), (s_{1/2}, p_{3/2}, \text{etc}), \\ & j = \tilde{l} - \frac{1}{2}, \text{aligned spin}(k < 0) \\ \tilde{k}_2 \equiv +(\tilde{l} + 1) & \text{if } \left(j = \tilde{l} + \frac{1}{2} \right), (p_{1/2}, d_{3/2}, \text{etc}), \\ & j = \tilde{l} + \frac{1}{2}, \text{unaligned spin}(k > 0) \end{cases} \quad (47)$$

and

$$k = \begin{cases} k_1 \equiv -(l+1) \text{ if } -(j+1/2), (s_{1/2}, p_{3/2}, \text{etc}), \\ \quad j = l + \frac{1}{2}, \text{aligned spin } (k \langle 0) \\ k_2 \equiv +l \text{ if } \left(j = l + \frac{1}{2} \right), (p_{1/2}, d_{3/2}, \text{etc}), \\ \quad j = l - \frac{1}{2}, \text{unaligned spin } (k \rangle 0) \end{cases} \quad (48)$$

With $\tilde{k}(\tilde{k}-1) = \tilde{l}(\tilde{l}+1)$ and $k(k-1) = l(l+1)$, which allows us to form two diagonal (3×3) matrixes $\hat{H}_{so-ph}(k_1, k_2)$ and $\hat{H}_{so-ph}(\tilde{k}_1, \tilde{k}_2)$, for (M.P.H.) potential, respectively, in (NC: 3D-RS) symmetries as

$$\begin{aligned} \left(\hat{H}_{so-ph} \right)_{11}(\tilde{k}_1) &= \tilde{k}_1 \Theta \left(\frac{a}{r^4} - \frac{b}{2r^3} \right) \\ \text{if } -(j+1/2), (s_{1/2}, p_{3/2}, \text{etc}), j &= \tilde{l} - \frac{1}{2}, \text{aligned spin } (k \langle 0) \\ \left(\hat{H}_{so-ph} \right)_{22}(\tilde{k}_2) &= \tilde{k}_2 \Theta \left(\frac{a}{r^4} - \frac{b}{2r^3} \right) \\ \text{if } \left(j = \tilde{l} + \frac{1}{2} \right), (p_{1/2}, d_{3/2}, \text{etc}), j &= \tilde{l} + \frac{1}{2}, \text{unaligned spin } (k \rangle 0) \\ \left(\hat{H}_{so-ph} \right)_{33} &= 0 \end{aligned} \quad (49)$$

and

$$\begin{aligned} \left(\hat{H}_{so-ph} \right)_{11}(k_1) &= k_1 \Theta \left(\frac{a}{r^4} - \frac{b}{2r^3} \right) \\ \text{if } -(j+1/2), (s_{1/2}, p_{3/2}, \text{etc}), j &= l + \frac{1}{2}, \text{aligned spin } (k \langle 0) \\ \left(\hat{H}_{so-ph} \right)_{22}(k_2) &= k_2 \Theta \left(\frac{a}{r^4} - \frac{b}{2r^3} \right) \\ \text{if } \left(j = l + \frac{1}{2} \right), (p_{1/2}, d_{3/2}, \text{etc}), j &= l - \frac{1}{2}, \text{unaligned spin } (k \rangle 0) \\ \left(\hat{H}_{so-ph} \right)_{33} &= 0 \end{aligned} \quad (50)$$

4.2. The exact relativistic spin-orbital spectrum for (M.P.H.) potential symmetries for n^{th} states for one-electron atoms in (NC: 3D-RSP) symmetry

In this subsection, we are going to study the modifications to the energy levels $(E_{nc-perd}(\Theta, \tilde{k}_1), E_{nc-peru}(\Theta, \tilde{k}_2))$ for $(-(j+1/2), (s_{1/2}, p_{3/2}, \text{etc}), j = \tilde{l} + \frac{1}{2})$, aligned spin $k \langle 0$ and

spin-down) and $(j = \tilde{l} + \frac{1}{2}, (p_{1/2}, d_{3/2}, \text{etc}), j = \tilde{l} - \frac{1}{2})$, an aligned spin $k \rangle 0$ and spin up), respectively, in first order of infinitesimal parameter Θ , for excited states n^{th} , for the spin symmetric case and the pseudo-spin spin-symmetry obtained by applying the standard perturbation theory, using Eqns. (24), (25) and (40) as

$$\begin{aligned} \int \Psi_{nk}^+(r, \theta, \phi) \left[\theta(E_{nc-ph}) \hat{Y}_{1per-ph}(r, \Theta, a, b) + \theta(-E_{nc-ph}) \hat{Y}_{2per-ph}(r, \Theta, a, b) \right] \Psi_{nk}(r, \theta, \phi) r^2 dr d\Omega \\ = \theta(E_{nc-ph}) \int F_{nk}^*(r) \hat{Y}_{1per-ph}(r, \Theta, a, b) F_{nk}(r) dr - \theta(-E_{nc-ph}) \int G_{nk}^*(r) \hat{Y}_{2per-ph}(r, \Theta, a, b) G_{nk}(r) dr \end{aligned} \quad (51)$$

The first part represent the modifications to the energy levels for the spin symmetric cases while the second part represent the modifications to the energy levels $(E_{nc-perd}(\Theta, \tilde{k}_1), E_{nc-peru}(\Theta, \tilde{k}_2))$ for the pseudo-spin spin-symmetry, then we have explicitly

$$E_{nc-perd}(\Theta, \tilde{k}_1) \equiv -\theta(E_{nc-ph}) \tilde{k}_1 \int G_{nk}^*(r) \left(\frac{a}{r^4} - \frac{b}{2r^3} \right) G_{nk}(r) dr \quad (52)$$

$$E_{nc-peru}(\Theta, \tilde{k}_2) \equiv \theta(-E_{nc-ph}) \tilde{k}_2 \int G_{nk}^*(r) \left(\frac{a}{r^4} - \frac{b}{2r^3} \right) G_{nk}(r) dr \quad (53)$$

Using Eqns. (25), (52) and (23), an explicit equation with the modified pseudo-harmonic conditions obtained as

$$\begin{aligned} E_{nc-perd}(\Theta, \tilde{k}_1) &\equiv -\theta(-E_{nc-ph}) \tilde{k}_1 \Theta \left(\frac{\Gamma\left(\nu + \frac{3}{2}\right)}{\Gamma\left(n + \nu + \frac{3}{2}\right)} \right)^2 \\ &\int_0^{+\infty} r^{2\nu+2} e^{-\mu r^2} \left[L_n^{\left(\nu + \frac{1}{2}\right)}(r^2) \right]^2 \left(\frac{a}{r^4} - \frac{b}{2r^3} \right) dr \end{aligned} \quad \dots (54)$$

$$E_{nc-per:u}(\Theta, \tilde{k}_2) \equiv -\theta(-E_{nc-ph}) \tilde{k}_2 \Theta \left(\frac{\tilde{N}n! \Gamma\left(\nu + \frac{3}{2}\right)}{\Gamma\left(n + \nu + \frac{3}{2}\right)} \right)^2 \int_0^{+\infty} r^{2\nu+2} e^{-\mu r^2} \left[L_{\nu}^{\left(\nu + \frac{1}{2}\right)}(r^2) \right]^2 \left(\frac{a}{r^4} - \frac{b}{2r^3} \right) dr \quad (55)$$

Now defining the new variable $X = r^2$ and the function we get

$$E_{nc-per:d}(\Theta, \tilde{k}_1) \equiv -\frac{1}{2} \theta(-E_{nc-ph}) \tilde{k}_1 \Theta \left(\frac{\tilde{N}n! \Gamma\left(\nu + \frac{3}{2}\right)}{\Gamma\left(n + \nu + \frac{3}{2}\right)} \right)^2 \int_0^{+\infty} X^{\nu - \frac{1}{2}} e^{-\mu X} \left[L_{\nu}^{\left(\nu + \frac{1}{2}\right)}(X) \right]^2 \left(\frac{a}{X^2} - \frac{b}{2X^{\frac{3}{2}}} \right) dX \quad (56)$$

$$E_{nc-per:u}(\Theta, \tilde{k}_2) \equiv -\frac{1}{2} \theta(-E_{nc-ph}) \tilde{k}_2 \Theta \left(\frac{\tilde{N}n! \Gamma\left(\nu + \frac{3}{2}\right)}{\Gamma\left(n + \nu + \frac{3}{2}\right)} \right)^2 \int_0^{+\infty} X^{\nu - \frac{1}{2}} e^{-\mu X} \left[L_{\nu}^{\left(\nu + \frac{1}{2}\right)}(X) \right]^2 \left(\frac{a}{X^2} - \frac{b}{2X^{\frac{3}{2}}} \right) dX \quad (57)$$

A direct simplification gives

$$E_{nc-per:d}(\Theta, \tilde{k}_1) \equiv -\frac{1}{2} \theta(-E_{nc-ph}) \tilde{k}_1 \Theta \left(\frac{\tilde{N}n! \Gamma\left(\nu_1 + \frac{3}{2}\right)}{\Gamma\left(n + \nu_1 + \frac{3}{2}\right)} \right)^2 (T_{1-ph}(D_0, r_0, \nu_1, n) + T_{2-ph}(D_0, r_0, \nu_1, n)) \quad (58)$$

$$E_{nc-per:u}(\Theta, \tilde{k}_2) \equiv -\frac{1}{2} \theta(-E_{nc-ph}) \tilde{k}_2 \Theta \left(\frac{\tilde{N}n! \Gamma\left(\nu_1 + \frac{3}{2}\right)}{\Gamma\left(n + \nu_1 + \frac{3}{2}\right)} \right)^2 (L_{1-ph}(D_0, r_0, \nu_2, n) + L_{2-ph}(D_0, r_0, \nu_2, n)) \quad (59)$$

Where, the four terms $T_{1-ph}(D_0, r_0, \nu_1, n)$, $T_{2-ph}(D_0, r_0, \nu_1, n)$, $L_{1-ph}(D_0, r_0, \nu_2, n)$ and $L_{2-ph}(D_0, r_0, \nu_2, n)$ are given by, respectively, as

$$T_{1-ph}(D_0, r_0, \nu_1, n) = a \int_0^{+\infty} X^{\nu_1 - \frac{5}{2}} e^{-\mu X} \left[L_{\nu_1}^{\left(\nu_1 + \frac{1}{2}\right)}(X) \right]^2 dX$$

$$T_{2-ph}(D_0, r_0, \nu_1, n) = -\frac{b}{2} \int_0^{+\infty} X^{\nu_1 - 2} e^{-\mu X} \left[L_{\nu_1}^{\left(\nu_1 + \frac{1}{2}\right)}(X) \right]^2 dX \quad (60a)$$

and

$$L_{1-ph}(D_0, r_0, \nu_1, n) = a \int_0^{+\infty} X^{\nu_2 - \frac{5}{2}} e^{-\mu X} \left[L_{\nu_2}^{\left(\nu_2 + \frac{1}{2}\right)}(X) \right]^2 dX$$

$$L_{2-ph}(D_0, r_0, \nu_1, n) = -\frac{b}{2} \int_0^{+\infty} X^{\nu_2 - 2} e^{-\mu X} \left[L_{\nu_2}^{\left(\nu_2 + \frac{1}{2}\right)}(X) \right]^2 dX \quad (60b)$$

Now we apply the special integral [77]

$$\int_0^{\infty} e^{-x\left(s + \frac{a_1 + a_2}{2}\right)} x^{\alpha + \beta} L_k^{\alpha}(a_1 x) L_k^{\alpha}(a_2 x) dx = \frac{\Gamma(1 + \alpha + \beta) \Gamma(1 + \alpha + k)}{k! k! \Gamma(1 + \alpha)} \left\{ \frac{d^k}{dh^k} \left[\frac{F\left(\frac{1 + \alpha + \beta}{2}, 1 + \frac{\alpha + \beta}{2}, 1 + \alpha; \frac{A^2}{B^2}\right)}{(1-h)^{1 + \alpha} B^{1 + \alpha + \beta}} \right] \right\}_{h=0} \quad (61)$$

Where, $A^2 = \frac{4a_1 a_2 h}{(1-h)^2}$, $B = s + \frac{a_1 + a_2}{2} \frac{1+h}{1-h}$,

$R\left(s + \frac{a_1 + a_2}{2}\right) > 0$, $a_1 > 0$, $a_2 > 0$ and $R(\alpha + \beta) > -1$,

which allow us to obtaining

$$T_{1-ph}(D_0, r_0, \nu_1, n) = D_0 r_0^{-2} \frac{\Gamma\left(\nu_1 - \frac{3}{2}\right) \Gamma\left(\frac{3}{2} + \nu_1 + n\right)}{n! n! \Gamma\left(\frac{3}{2} + \nu_1\right)} \left\{ \frac{d^n}{dh^n} \left[\frac{F\left(\frac{\nu_1 - 3/2}{2}, 1 + \frac{\nu_1 - 3/2}{2}, \nu_1 + \frac{3}{2}; \frac{A^2}{B^2}\right)}{(1-h)^{\nu_1 + \frac{3}{2}} B_1^{\nu_1 - 3/2}} \right] \right\}_{h=0} \quad (62)$$

and

$$T_{2-ph}(D_0, r_0, \nu_1, n) = -\frac{D_0 r_0^{+2}}{2} \frac{\Gamma(\nu_1) \Gamma\left(\frac{3}{2} + \nu_1 + n\right)}{n! n! \Gamma\left(\frac{3}{2} + \nu_1\right)} \dots$$

$$\left\{ \frac{d^n}{dh^n} \left[\frac{F\left(\frac{\nu_1}{2}\right), 1 + \frac{\nu_1 - 1}{2}; \nu_1 + \frac{3}{2}; \frac{A^2}{B^2} \right]}{(1-h)^{\nu_1 + \frac{3}{2}} B^{\nu_1}} \right\}_{h=0}$$

(63)

Here

$$\nu_1(\nu_1 + 1) = \tilde{k}_1(\tilde{k}_1 + 1) + (M + E - C)D_0 r_0^{-2},$$

$$A^2 = \frac{4h}{(1-h)^2} \quad B = \mu - 1 + \frac{1+h}{1-h},$$

$$L_{1-ph}(D_0, r_0, \nu_2, n) = T_{1-ph}(D_0, r_0, \nu_1 \rightarrow \nu_2, n) \text{ and}$$

$$L_{2-ph}(D_0, r_0, \nu_2, n) = T_{2-ph}(D_0, r_0, \nu_1 \rightarrow \nu_2, n), \text{ which gives explicitly}$$

$$L_{1-ph}(D_0, r_0, \nu_2, n) = D_0 r_0^{-2} \frac{\Gamma\left(\nu_2 - \frac{3}{2}\right) \Gamma\left(\frac{3}{2} + \nu_2 + n\right)}{n! n! \Gamma\left(\frac{3}{2} + \nu_2\right)}$$

$$\left\{ \frac{d^n}{dh^n} \left[\frac{F\left(\frac{\nu_2 - 3/2}{2}\right), 1 + \frac{\nu_2 - 3/2}{2}; \nu_2 + \frac{3}{2}; \frac{A^2}{B^2} \right]}{(1-h)^{\nu_2 + \frac{3}{2}} B_1^{\nu_2 - 3/2}} \right\}_{h=0}$$

(64)

and

$$L_{2-ph}(D_0, r_0, \nu_2, n) = -\frac{D_0 r_0^{+2}}{2} \frac{\Gamma(\nu_2) \Gamma\left(\frac{3}{2} + \nu_2 + n\right)}{n! n! \Gamma\left(\frac{3}{2} + \nu_2\right)}$$

$$\left\{ \frac{d^n}{dh^n} \left[\frac{F\left(\frac{\nu_2}{2}\right), 1 + \frac{\nu_2 - 1}{2}; \nu_2 + \frac{3}{2}; \frac{A^2}{B^2} \right]}{(1-h)^{\nu_2 + \frac{3}{2}} B^{\nu_2}} \right\}_{h=0}$$

(65)

Where, $\nu_2(\nu_2 + 1) = \tilde{k}_2(\tilde{k}_2 + 1) + (M + E - C)D_0 r_0^{-2}$. Substituting Eqns. (64) and (65) and (62) and (63) into Eqns. (58) and (59), respectively, we obtain the modifications to the energy levels $(E_{nc-per.d}(\Theta, \tilde{k}_1), E_{nc-per.u}(\Theta, \tilde{k}_2))$ produced by relativistic spin-orbital effect under the pseudo spin symmetry conditions. Knowing the energy levels

$(E_{nc-per.d}(\Theta, k_1), E_{nc-per.u}(\Theta, k_2))$ produced by relativistic spin-orbital effect under the spin symmetry conditions, it can be determined by means of same procedures as before and avoid repetition we just make the following steps

$$\tilde{N} \rightarrow N, \tilde{k}_1 \rightarrow k_1, \tilde{k}_2 \rightarrow k_2 \text{ and } \theta(-E_{nc-ph}) \rightarrow -\theta(E_{nc-ph})$$

(66)

Which implies that

$(E_{nc-per.d}(\Theta, k_1), E_{nc-per.u}(\Theta, k_2))$ can be expressed, respectively, as

$$E_{nc-per.d}(\Theta, k_1) \equiv \frac{1}{2} \theta(E_{nc-ph}) k_1 \Theta \left[N n! \frac{\Gamma\left(\nu(k_2) + \frac{3}{2}\right)}{\Gamma\left(n + \nu(k_2) + \frac{3}{2}\right)} \right]^2$$

$$(T_{1-ph}(D_0, r_0, \nu(k_2), n) + T_{2-ph}(D_0, r_0, \nu(k_2), \nu, n))$$

(67)

$$E_{nc-per.u}(\Theta, k_2) \equiv \frac{1}{2} \theta(E_{nc-ph}) k_2 \Theta \left[N n! \frac{\Gamma\left(\nu(k_1) + \frac{3}{2}\right)}{\Gamma\left(n + \nu(k_1) + \frac{3}{2}\right)} \right]^2$$

$$(L_{1-ph}(D_0, r_0, \nu(k_1), n) + L_{2-ph}(D_0, r_0, \nu(k_1), n))$$

(68)

The negative and positive signs of the coefficients $\theta(-E_{nc-ph})$ and $\theta(E_{nc-ph})$ are necessary to ensure that the modifications to the energy levels under the pseudo spin symmetry conditions and spin symmetry conditions are negative and positive, respectively.

4.3. The exact relativistic magnetic spectrum for (M.P.H.) potential for excited n^{th} states for one-electron atoms in (NC: 3D- R S) symmetries

Having obtained the exact modifications to the energy levels $(E_{nc-per.d}(\Theta, \tilde{k}_1), E_{nc-per.u}(\Theta, \tilde{k}_2))$ and $(E_{nc-per.d}(\Theta, k_1), E_{nc-per.u}(\Theta, k_2))$ under the pseudo spin symmetry conditions and spin symmetry conditions, respectively, for excited n^{th} states, produced by noncommutative spin-orbital Hamiltonian operator, we now consider another interested physically meaningful phenomena, which is also produced from the perturbative terms of pseudoharmonic potential related to the influence of an external uniform magnetic field, it's

sufficient to apply the following two replacements to describing these phenomena

$$D_0 \left(\frac{r_0^{-2}}{r^4} - \frac{r_0^{+2}}{2r^3} \right) \vec{L} \vec{\Theta} \rightarrow \chi D_0 \left(\frac{r_0^{-2}}{r^4} - \frac{r_0^{+2}}{2r^3} \right) \vec{B} \vec{L}$$

or $\chi D_0 \left(\frac{r_0^{-2}}{r^4} - \frac{r_0^{+2}}{2r^3} \right) \vec{B} \vec{L}$

and $\Theta \rightarrow \chi B$ (69)

Here χ is infinitesimal real proportional's constants, and we choose the magnetic field $\vec{B} = B \vec{k}$, which allow us to introduce the modified new magnetic Hamiltonian $\hat{H}_{mag-ph}(r, a, b, \chi)$ in (NC: 3D-RS), as:

$$\hat{H}_{mag-ph}(D_0, r_0, \chi) = \begin{cases} \chi D_0 \left(\frac{r_0^{-2}}{r^4} - \frac{r_0^{+2}}{2r^3} \right) \vec{B} \vec{L} & \text{for pseudospinsymmetry} \\ \chi D_0 \left(\frac{r_0^{-2}}{r^4} - \frac{r_0^{+2}}{2r^3} \right) \vec{B} \vec{L} & \text{for spinsymmetry} \end{cases} \quad (70)$$

Here $(-\vec{S} \vec{B})$ denote to the ordinary Hamiltonian of Zeeman Effect. To obtain the exact noncommutative magnetic modifications of energy $E_{mag-ph}(\chi, n, \tilde{m}, D_0, r_0)$ and $E_{mag-ph}(\chi, n, m, D_0, r_0)$ for modified pseudo-harmonic potential, under the pseudo spin symmetry conditions and spin symmetry conditions, respectively, which is produced automatically by the effect of $\hat{H}_{m-ph}(r, D_0, r_0, \chi)$, we make the following two simultaneously replacements

$$\tilde{k}_1 \rightarrow \tilde{m}, k_1 \rightarrow m \text{ and } \Theta \rightarrow \chi B \quad (71)$$

Then, the relativistic magnetic modification $E_{mag-ph}(\chi, n, \tilde{m}, D_0, r_0)$ and $E_{mag-ph}(\chi, n, m, D_0, r_0)$ corresponding n^{th} excited states, in (NC-3D: RS) symmetries can be determined from the following relation

$$E_{mag-ph}(\chi, n, \tilde{m}, D_0, r_0) = -\frac{1}{2} \theta(-E_{nc-ph}) \chi B \left(\frac{\tilde{N} n! \frac{\Gamma\left(v_1 + \frac{3}{2}\right)}{\Gamma\left(n + v_1 + \frac{3}{2}\right)}}{\left(L_{1-ph}(D_0, r_0, v_2, n) + L_{2-ph}(D_0, r_0, v_2, n)\right) \tilde{m}} \right) \quad (72)$$

and

$$E_{mag-ph}(\chi, n, m, D_0, r_0) = \frac{1}{2} \theta(E_{nc-ph}) \chi B \left(\frac{N n! \frac{\Gamma\left(v(k_1) + \frac{3}{2}\right)}{\Gamma\left(n + v(k_1) + \frac{3}{2}\right)}}{\left(L_{1-ph}(D_0, r_0, v(k_1), n) + L_{2-ph}(D_0, r_0, v(k_1), n)\right) m} \right) \quad (73)$$

Where, \tilde{m} and m denote to the angular momentum quantum numbers $-\tilde{l} \leq \tilde{m} \leq \tilde{l}$ and $-l \leq m \leq l$, which allows us to fixing $(2\tilde{l} + 1)$ and $(2l + 1)$ values, respectively.

5. The exact modified global spectrum for (M.P.H.) potential in (NC-3D: RS) symmetries for one-electron atoms

Let us now resume the n^{th} excited states eigen-energies $(E_{nc-pd}(\Theta, \tilde{k}_1, \chi, n, \tilde{m}, r_0, D_0))$, $E_{nc-pu}(\Theta, \tilde{k}_2, \chi, n, \tilde{m}, r_0, D_0)$ and $(E_{nc-d}(\Theta, k_1, \chi, n, m, r_0, D_0), E_{nc-u}(\Theta, k_2, \chi, n, m, r_0, D_0))$ of modified Dirac equation corresponding to pseudo spin symmetry conditions and spin symmetry conditions, respectively, in the first order of parameter Θ , for (M.P.H.) potential in (NC: 3D-RS) symmetries based to obtained new results Eqns. (58), (59), (67), (68), (72) and (73), in addition to the original results Eqns. (20) and (22) of energies in commutative space, we obtain the following original results

$$E_{nc-pd}(\Theta, \tilde{k}_1, \chi, n, \tilde{m}, r_0, D_0) = E_{n\tilde{k}_1} - \frac{1}{2} \theta(-E_{nc-ph}) \tilde{k}_1 \Theta \left(\frac{\tilde{N} n! \frac{\Gamma\left(v_1 + \frac{3}{2}\right)}{\Gamma\left(n + v_1 + \frac{3}{2}\right)}}{\left(T_{1-ph}(D_0, r_0, v_1, n) + T_{2-ph}(D_0, r_0, v_1, n)\right) - \frac{1}{2} \theta(-E_{nc-ph}) \chi B \left(\frac{\tilde{N} n! \frac{\Gamma\left(v_1 + \frac{3}{2}\right)}{\Gamma\left(n + v_1 + \frac{3}{2}\right)}}{\left(L_{1-ph}(D_0, r_0, v_2, n) + L_{2-ph}(D_0, r_0, v_2, n)\right) \tilde{m}} \right)} \right) \quad (74)$$

$$\begin{aligned}
 E_{nc-ph}(\Theta, \tilde{k}_2, \chi, n, \tilde{m}, r_0, D_0) &= E_{n\tilde{k}_2} \frac{1}{2} \theta(-E_{nc-ph}) \tilde{k}_2 \Theta \left(\frac{\Gamma\left(\nu_1 + \frac{3}{2}\right)}{\Gamma\left(n + \nu_1 + \frac{3}{2}\right)} \right)^2 \\
 (L_{1-ph}(D_0, r_0, \nu_2, n) + L_{2-ph}(D_0, r_0, \nu_2, n)) &- \frac{1}{2} \theta(-E_{nc-ph}) \chi \mathbb{B} \left(\frac{\Gamma\left(\nu_1 + \frac{3}{2}\right)}{\Gamma\left(n + \nu_1 + \frac{3}{2}\right)} \right) \\
 (L_{1-ph}(D_0, r_0, \nu_2, n) + L_{2-ph}(D_0, r_0, \nu_2, n)) \tilde{m} &
 \end{aligned} \tag{75}$$

$$\begin{aligned}
 E_{nc-d}(\Theta, k_1, \chi, n, m, r_0, D_0) &= E_{nk_1} + \frac{1}{2} \theta(E_{nc-ph}) k_1 \Theta \left(\frac{\Gamma\left(\nu(k_2) + \frac{3}{2}\right)}{\Gamma\left(n + \nu(k_2) + \frac{3}{2}\right)} \right)^2 \\
 (T_{1-ph}(D_0, r_0, \nu(k_2), n) + T_{2-ph}(D_0, r_0, \nu(k_2), \nu, n)) &+ \frac{1}{2} \theta(E_{nc-ph}) \chi \mathbb{B} \left(\frac{\Gamma\left(\nu(k_1) + \frac{3}{2}\right)}{\Gamma\left(n + \nu(k_1) + \frac{3}{2}\right)} \right) \\
 (L_{1-ph}(D_0, r_0, \nu(k_1), n) + L_{2-ph}(D_0, r_0, \nu(k_1), n)) m &
 \end{aligned} \tag{76}$$

$$\begin{aligned}
 E_{nc-u}(\Theta, k_2, \chi, n, m, r_0, D_0) &= E_{nk_2} + \frac{1}{2} \theta(E_{nc-ph}) k_2 \Theta \left(\frac{\Gamma\left(\nu(k_1) + \frac{3}{2}\right)}{\Gamma\left(n + \nu(k_1) + \frac{3}{2}\right)} \right)^2 \\
 (L_{1-ph}(D_0, r_0, \nu(k_1), n) + L_{2-ph}(D_0, r_0, \nu(k_1), n)) &+ \frac{1}{2} \theta(E_{nc-ph}) \chi \mathbb{B} \left(\frac{\Gamma\left(\nu(k_1) + \frac{3}{2}\right)}{\Gamma\left(n + \nu(k_1) + \frac{3}{2}\right)} \right) \\
 (L_{1-ph}(D_0, r_0, \nu(k_1), n) + L_{2-ph}(D_0, r_0, \nu(k_1), n)) m &
 \end{aligned} \tag{77}$$

Now, it is important to construct Hamiltonian operator \hat{H}_{nc-ph} for (M.P.H.) potential based on previously obtained results. Naturally, to consider the first term in the modified Hamiltonian operator represents the kinetic energy and the potential energy in ordinary commutative space \hat{H}_{0ph} of the fermionic particle that was presented by Eqn. (14), the second term $\hat{H}_{so-ph}(k_1, k_2)$ or $\hat{H}_{so-ph}(\tilde{k}_1, \tilde{k}_2)$ represents the induced spin-orbital parts for the pseudo spin symmetry conditions and spin symmetry, and the last term is the modified new magnetic Hamiltonian $\hat{H}_{mag-m}(r, a, b, \chi)$

$$\hat{H}_{nc-ph} = \hat{H}_{0ph} + \begin{cases} \hat{H}_{so-ph}(\tilde{k}_1, \tilde{k}_2) + \chi \mathcal{D}_0 \left(\frac{r_0^{-2}}{r^4} - \frac{r_0^{+2}}{2r^3} \right) \vec{B}\vec{L} \\ \text{for pseudo spin symmetry} \\ \hat{H}_{so-ph}(k_1, k_2) + \chi \mathcal{D}_0 \left(\frac{r_0^{-2}}{r^4} - \frac{r_0^{+2}}{2r^3} \right) \vec{B}\vec{L} \\ \text{for spin symmetry} \end{cases} \tag{78}$$

In this way, one can obtain the complete energy spectra for (M.P.H.) potential in (NC: 3D-RS) symmetries. Know the following accompanying constraint relations:

- The original spectrum contains two possible values of energies in ordinary three dimensional space which represented by Eqns. (13) and (47).

As mentioned in the previous subsection, the quantum numbers \tilde{m} and m satisfied the two intervals: $-\tilde{l} \leq \tilde{m} \leq +\tilde{l}$ and $-l \leq m \leq +l$, thus we have $(2\tilde{l} + 1)$ and $(2l + 1)$ values, respectively.

- We have also two values for $(j = \tilde{l} + \frac{1}{2}$ and $j = \tilde{l} - \frac{1}{2})$ and $(j = l + \frac{1}{2}$ and $j = l - \frac{1}{2})$ for pseudo spin symmetry conditions and spin symmetry.

Allow us to deduce the important original results: every state in usually three dimensional space will be replaced by $2(2\tilde{l} + 1)$ and $2(2l + 1)$ sub-states and then the degenerated state can take

$$2 \sum_{i=0}^{n-1} (2l + 1) \equiv 2n^2 \text{ values in (NC: 3D-RS)}$$

symmetries. Finally, we resume our original results in this article, the first one is the induced pseudospin-orbital and spin-orbital Hamiltonian operators $(\hat{H}_{so-ph}(\tilde{k}_1, \tilde{k}_2))$ and $(\hat{H}_{so-ph}(k_1, k_2))$

and corresponding eigenvalues $(E_{nc-perd}(\Theta, \tilde{k}_1), E_{nc-peru}(\Theta, \tilde{k}_2))$ and $(E_{nc-perd}(\Theta, k_1), E_{nc-peru}(\Theta, k_2))$, respectively as

$$\hat{H}_{so-ph}(\tilde{k}_1, \tilde{k}_2)\Psi_{nk}(r, \theta, \varphi) \Rightarrow \begin{cases} \left(\hat{H}_{so-ph}\right)_{11}(\tilde{k}_1)\left(i\frac{G_{n\tilde{k}}(r)}{r}Y_{jm}^{\tilde{l}}(\theta, \varphi)\right) \\ = E_{nc-per;d}(\Theta, \tilde{k}_1)\left(i\frac{G_{n\tilde{k}}(r)}{r}Y_{jm}^{\tilde{l}}(\theta, \varphi)\right) \\ \left(\hat{H}_{so-ph}\right)_{22}(\tilde{k}_2)\left(i\frac{G_{n\tilde{k}}(r)}{r}Y_{jm}^{\tilde{l}}(\theta, \varphi)\right) \\ = E_{nc-per;u}(\Theta, \tilde{k}_1)\left(i\frac{G_{n\tilde{k}}(r)}{r}Y_{jm}^{\tilde{l}}(\theta, \varphi)\right) \\ \left(\hat{H}_{so-ph}\right)_{33}(\tilde{k}_1)\left(i\frac{G_{n\tilde{k}}(r)}{r}Y_{jm}^{\tilde{l}}(\theta, \varphi)\right) = 0 \end{cases} \quad (79)$$

and

$$\hat{H}_{so-ph}(k_1, k_2)\Psi_{nk}(r, \theta, \varphi) \Rightarrow \begin{cases} \left(\hat{H}_{so-ph}\right)_{11}(k_1)\left(\frac{F_{nk}(r)}{r}Y_{jm}^l(\theta, \varphi)\right) \\ = E_{nc-per;d}(\Theta, k_1)\left(\frac{F_{nk}(r)}{r}Y_{jm}^l(\theta, \varphi)\right) \\ \left(\hat{H}_{so-ph}\right)_{22}(k_2)\left(\frac{F_{nk}(r)}{r}Y_{jm}^l(\theta, \varphi)\right) \\ = E_{nc-per;u}(\Theta, k_2)\left(\frac{F_{nk}(r)}{r}Y_{jm}^l(\theta, \varphi)\right) \\ \left(\hat{H}_{so-ph}\right)_{33}(k_1, k_2)\left(\frac{F_{nk}(r)}{r}Y_{jm}^l(\theta, \varphi)\right) = 0 \end{cases} \quad (80)$$

The second original result is the induced modified new magnetic Hamiltonian operator $\hat{H}_{mag-mt}(r, a, b, \chi)$ and corresponding eigenvalues $E_{mag-ph}(\chi, n, \tilde{m}, D_0, r_0)$ and $E_{mag-ph}(\chi, n, m, D_0, r_0)$ are respectively as

$$\hat{H}_{mag-mt}(r, a, b, \chi)\Psi_{nk}(r, \theta, \varphi) \equiv \frac{\chi D_0}{r} \begin{pmatrix} r_0^{-2} & -r_0^{+2} \\ r^4 & 2r^3 \end{pmatrix} \begin{pmatrix} \tilde{B}\tilde{L} & 0 \\ 0 & \tilde{B}\tilde{L} \end{pmatrix} \begin{pmatrix} F_{nk}(r)Y_{jm}^l(\theta, \varphi) \\ iG_{n\tilde{k}}(r)Y_{jm}^{\tilde{l}}(\theta, \varphi) \end{pmatrix} \\ = \begin{pmatrix} E_{magph}(\chi, n, m, D_0, r_0)\frac{F_{nk}(r)}{r}Y_{jm}^l(\theta, \varphi) \\ E_{magph}(\chi, n, \tilde{m}, D_0, r_0)i\frac{G_{n\tilde{k}}(r)}{r}Y_{jm}^{\tilde{l}}(\theta, \varphi) \end{pmatrix} \quad (81)$$

It is worth mentioning that in the limit $\Theta \rightarrow 0$ we obtain the commutative result.

6. Concluding Remarks

In this study, we have computed the exact analytical bound state solutions, the energy spectra and the corresponding noncommutative Hermitian Hamiltonian operator for three dimensional Dirac equations in spherical coordinates for (M.P.H.) potential by using generalization Boopp's Shift method and standard perturbation theory. It is found that the energy eigenvalues depend on the

dimensionality of the problem and new atomic quantum numbers

$(j = \tilde{l} \pm 1/2, j = l \pm 1/2, \tilde{s} = \pm 1/2, l, \tilde{l})$ and the angular momentum quantum number in addition to the infinitesimal parameter Θ in the symmetries of (NC: 3D-RSP). We also showed that the obtained energy spectra is degenerate and every old state will be replaced by $2(\tilde{l} + 1)$ and $2(l + 1)$ sub-states under the pseudo spin symmetry conditions and spin symmetry conditions, respectively, for excited n^{th} states. In the limit when $\Theta \rightarrow 0$, we recover the ordinary results of commutative space.

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