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Schrödinger Equation with Double Cosine- and Sine-Squared Potential by Darboux Transformation Method and Super-symmetry

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In this paper, we consider one-dimensional Schrödinger equation for the double cosine- and sine-squared potential. We construct the first order Darboux transformation and the real valued condition of the transformed potential for two corresponding equations. In this case, we obtain the transformed potential and wave function, and finally investigate the super-symmetry aspect of such corresponding equation. Also, we show that the first order equation is satisfied by commutative and anti-commutative algebra with the constant condition at different limits for *x*.

1. Introduction

There are several methods to study the integrability model. One of the methods that we focus here is Darboux transformation. It is well known that Darboux transformation [1] is one of the major tools for the analysis of physical systems and for finding new solvable systems, using a linear differential operator. Darboux constructs solutions of one ordinary differential equation in terms of another ordinary differential equation. It has been shown that the transformation method is useful in finding soliton solutions of the integrable systems [2-4] and constructing super-symmetric quantum mechanical systems [5-7]. Also, more general solvable cases were obtained by means of factorization methods [8] and via Lie algebraic approaches [9-13]. Darboux transformation is known as one of the most powerful methods for finding solvable Schrödinger equations with constant mass, in the context of which it is also called super-symmetric factorization method [14]. On the other hand during the past few years there has been great interest in studying class of trigonometric potentials [15]. The solution of such equation may be found by mapping it onto a Schrödinger-like equation. So, we take advantage of Darboux transformation and obtain the generalized form of double-cosine and sine-squared potential. The Darboux transformation has been extensively used in quantum mechanics in search of isospectral potential for Schrödinger equations of constant mass and position-dependent mass [16-21]. So, we take advantage from such transformation and obtain the effective potential, modified wave function, shape invariance condition and generators of supersymmetry algebra for the two corresponding potentials. This paper is organized as follows: we first introduce the one-dimensional Schrödinger equation for the double-cosine and sine-squared potential and apply such transformation to these equations. In that case, we show that the corresponding potential change to new form of potential. Finally, we study the supersymmetry version and shape invariance condition for transformed double-cosine and sine-squared potential.

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2. Double-Cosine potential

First of all we are going to consider a single particle in double-cosine potential which is given by [22],

$$V(x) = \begin{cases} V_1 \cos(x) + V_2 \cos(2x) & 0 < x < 2\pi \\ \infty & x < 0 \; and \; x > 2\pi \end{cases} \tag{1}$$

Where Schrödinger like equation will be as,

$$\left[-\frac{d^2}{dx^2} + V_1 \cos(x) + V_2 \cos(2x)\right]\psi(x) = E\psi(x), \qquad \psi(0) = \psi(2\pi) = 0 \tag{2}$$

The maximum of the potential (1) occurs within the given interval between $x = 0$ and $x = 2\pi$ and has the value of $V_{max} = V_1 + V_2$, while the minimum occurs at $x = \pi$ with the value of $V_{max} = V_2 - V_1$. Clearly, if $V_1 = 0$ and $V_2 > 0$, the problem corresponds to Mathieus equation [23]. Assume the general solution of the differential equation (2) is satisfied the boundary conditions takes the form: $\psi(x) = \sin\left(\frac{\pi}{2}\right) f(x)$. In that case, we use such condition and make the second order equation (2) in terms of $f(x)$, which is given by

$$f''(x) + cot\left(\frac{x}{2}\right) f'(x) - \left[\frac{1}{4} + V_1 \cos(x) + 2V_2 cos^2(x) - V_2 - E\right] f(x) = 0 \tag{3}$$

Now we choose the following variable,

$$f(x) = exp\left(-\frac{V_1}{2}\right) g(y), \qquad y = cos\left(\frac{x}{2}\right) \tag{4}$$

And we obtain,

$$(1 - y^2)g''(y) + \left[y\left(1 + 4V_1(1 - y^2)\right)\right]g'(y) - [1 - 2V_1 + 4V_2 - 4E - 4(V_1^2 + 8V_2)y^2(1 - y^2)]g(y) = 0 \tag{5}$$

So, the exact solution for the E_0 and V_2 are

$$g_0(x) = 1\,, \qquad E_0 = \frac{1}{4} - \frac{V_1}{2} + V_2\,, \qquad V_2 = -\frac{1}{8}V_1^2 \tag{6}$$

In order to change the equation (2) into the form with known polynomial, we need to choose the following variable

$$g(y) = u(y)P(y) \tag{7}$$

So, one can rewrite the equation (5) as

$$\begin{aligned}(1 - y^2)P''(y) &+ \left[2(1 - y^2)\frac{u'}{u} + y\left(1 + 4V_1(1 - y^2)\right)\right] P'(y) \\ &+ \Bigg[(1 - y^2)\frac{u''}{u} + y\left(1 + 4V_1(1 - y^2)\right)\frac{u'}{u} \\ &- \{1 - 2V_1 + 4V_2 - 4E - 4(V_1^2 + 8V_2)y^2(1 - y^2)\}\Bigg] P(y) = 0\end{aligned} \tag{8}$$

Here, we consider the following associated - Legendre differential equation [24-26]

$$\begin{aligned}(1 - y^2)P''^{\alpha,\beta}_{n,m}(y) &- [\alpha - \beta + (\alpha + \beta + 2)y]P'^{\alpha,\beta}_{n,m}(y) \\ &+ \left[n(\alpha + \beta + n + 1) - \frac{m(\alpha + \beta + m) + m(\alpha - \beta)y}{1 - y^2}\right] P^{\alpha,\beta}_{n,m}(y) = 0\end{aligned} \tag{9}$$

Also, we compare the equations (8) and (9) to each other and obtain the wave function $u(y)$ and $g(y)$ as

$$u(y) = e^{-V_1 y^2}\left(\frac{1+y}{1-y}\right)^{\frac{\beta-\alpha}{4}}(1-y^2)^{\frac{\alpha+\beta+3}{4}} \tag{10}$$

So, the general form of $g(y)$ and $g(x)$ functions will be following

$$g(y) = e^{-V_1 y^2}\left(\frac{1+y}{1-y}\right)^{\frac{\beta-\alpha}{4}}(1-y^2)^{\frac{\alpha+\beta-1}{4}}P_{n,m}^{\alpha,\beta}\ , \qquad y = cos\left(\frac{x}{2}\right),$$

$$g(x) = e^{-V_1 cos^2\left(\frac{x}{2}\right)}\left[\frac{1+cos\left(\frac{x}{2}\right)}{1-cos\left(\frac{x}{2}\right)}\right]^{\left(\frac{\beta-\alpha}{4}\right)} sin\left(\frac{x}{2}\right)^{\left(\frac{\alpha+\beta+3}{2}\right)} P_{n,m}^{\alpha,\beta}(x) \tag{11}$$

Also, we take advantage from comparing (8) and (9) and obtain the V_1, E and $f(x)$

$$V_1 = \frac{1}{4}\left(m-\alpha-\beta+\frac{3}{2}\right), \tag{12}$$

$$E = \frac{1}{4}\left[(\beta-\alpha)^2 - \left(\alpha+\beta+\frac{7}{4}\right) - \frac{1}{32}\left(m-\alpha-\beta+\frac{3}{2}\right)^2 - n(\alpha+\beta+n+1) + m\left(\alpha+\beta+m+\frac{1}{2}\right)\right] \tag{13}$$

And,

$$f(x) = e^{-\frac{1}{2}V_1\left(1+2cos^2\left(\frac{x}{2}\right)\right)}\left[\frac{1+cos\left(\frac{x}{2}\right)}{1-cos\left(\frac{x}{2}\right)}\right]^{\left(\frac{\beta-\alpha}{4}\right)} sin\left(\frac{x}{2}\right)^{\left(\frac{\alpha+\beta+3}{2}\right)} P_{n,m}^{\alpha,\beta}(x) \tag{14}$$

3. Trigonometric Sine-Squared Potential.

The second example we consider here is one-dimensional Schrödinger equation for the trigonometric sine-squared potential, which is given by

$$\left[-\frac{d^2}{dx^2} + V_0 sin^2\left(\frac{x}{a}\right)\right]C(x) = EC(x) \tag{15}$$

Where, $C\left(\frac{\pi a}{2}\right) = C\left(\frac{-\pi a}{2}\right)$

This boundary condition leads us to consider following change of variable

$$C(x) = cos\left(\frac{x}{a}\right)f(x) \tag{16}$$

So, one can rewrite equation (15) as

$$f''(x) = \frac{2}{a}tan\left(\frac{x}{2}\right)f'(x) + \left[\frac{1}{a^2} - E + V_0 sin^2\left(\frac{x}{a}\right)\right]f(x) \tag{17}$$

By putting $x = a\ Arcsin(y)$ in (17), one can obtain

$$f''(y) = \frac{3y}{1-y^2}f'(y) + \left(\frac{\omega}{1-y^2} - \mu\right)f(y) \tag{18}$$

Where $\omega = 1 + \mu - a^2$ and $\mu = V_0 a^2$. By choosing suitable variable the same as previous case we have

$$f(y) = u(y)P(y) \tag{19}$$

We substitute equation (19) in (18) and obtain the following equation

$$(1-y^2)P''(y)+\left[2(1-y^2)\frac{u'}{u}-3y\right]P'(y)+\left[(1-y^2)\frac{u''}{u}-3y\frac{u'}{u}-\omega+\mu(1-y^2)\right]P(y)=0 \tag{20}$$

Here, we compare equations (9) and (20) to each other, and arrive at the following expression for *u*(*y*), *f*(*y*) and *f*(*x*), respectively

$$u(y)=\left(\frac{1+y}{1-y}\right)^{\frac{\beta-\alpha}{4}}(1-y^2)^{\frac{\alpha+\beta-1}{4}} \tag{21}$$

$$f(y)=\left(\frac{1+y}{1-y}\right)^{\frac{\beta-\alpha}{4}}(1-y^2)^{\frac{\alpha+\beta-1}{4}}p_{n,m}^{\alpha,\beta}(y)\ ,\quad y=sin\frac{x}{a}$$

And

$$f(x)=\left(\frac{1+sin\left(\frac{x}{a}\right)}{1-sin\left(\frac{x}{a}\right)}\right)^{\frac{\beta-\alpha}{4}}cos\left(\frac{x}{a}\right)^{\frac{\alpha+\beta-1}{2}}p_{n,m}^{\alpha,\beta}(x) \tag{22}$$

On the other hand this comparison gives us opportunity to obtain the energy spectrum and the wave function, which are given by

$$E=\frac{\hbar^2}{2\mu a^2}[n(\alpha+\beta+n)+1] \tag{23}$$

And

$$C(x)=cos\left(\frac{x}{a}\right)\left(\frac{1+sin\left(\frac{x}{a}\right)}{1-sin\left(\frac{x}{a}\right)}\right)^{\frac{\beta-\alpha}{4}}cos\left(\frac{x}{a}\right)^{\frac{\alpha+\beta-1}{2}}p_{n,m}^{\alpha,\beta}(x) \tag{24}$$

The corresponding energy spectrum always is positive, so we have a stable system.

4. Darboux Transformation and Double-Cosine potential

Now we are going to apply the Darboux transformation to corresponding example such as double-cosine and trigonometric sine-squared potential. So, we simplify the equation (5) as,

$$Fg_{yy}+Gg_y-Vg=0 \tag{25}$$

Where *F* and *G* and *V* are, respectively

$$F=(1-y^2)\ ,\qquad G=y+4V_1y(1-y^2)$$
$$V=1-2V_1+4V_2-4E-4(V_1^2+8V_2)y^2(1-y^2) \tag{26}$$

Here we introduce the new variable η that plays an important role in Darboux transformation. So, we can write the above equation with η variable, which is given by

$$ig_t+\eta g=0\ ,\qquad \eta=F\partial_{yy}+G\partial_y-V \tag{27}$$

The Darboux transformation helps us to write the equations (25) and (27) in a new form with different potential as

$$i\hat{g}_t+\hat{\eta}g=0\ ,\quad \hat{\eta}=F\partial_{yy}+G\partial_y-\hat{V} \tag{28}$$

Where, $V\neq\hat{V}$, implies $g(y)\neq\hat{g}(y)$. We introduce transformation operator Δ as

$$\Delta(i\partial+\eta)=(i\partial+\hat{\eta})\Delta \tag{29}$$

These are called Darboux transformation operator for the Hamiltonian η and $\hat{\eta}$, respectively.

The operator Δ transforms any solution $f(y)$ into a new solution

$$\hat{f}(y)=\Delta f(y) \tag{30}$$

Let Darboux transformation operator be the form of a linear, first- order differential operator

$$\Delta = A + B\partial_y \tag{31}$$

Where, we take special case as *A* = *B*. In order to find *A* or *B*, we consider the explicit form of Δ and $\hat{\Delta}$ in form of the Darboux transformation and apply it to the solution *g*(*y*), so

$$\Delta(i\partial_t + \eta)g(y) = (i\partial_t + \hat{\eta})\Delta g(y) \tag{32}$$

Making linear independence of *g*(*y*) and its partial derivatives, we collect their respective coefficients and put them equal to zero, from which one can obtain the following value for the functions *A* and $\hat{V}$

$$2F = (f)_y B \quad \Rightarrow \quad B = \frac{-1}{y}(1 - y^2) \tag{33}$$

So, the Darboux transformation operator will be as

$$\Delta = \frac{-1}{y}(1 - y^2)\left(1 + \partial_y\right) \ or \ \Delta = -cot\left(\frac{x}{2}\right)\left[sin\left(\frac{x}{2}\right) - 2\frac{d}{dx}\right] \tag{34}$$

The relation between V and $\hat{V}$ will be as

$$\hat{V} = v + \frac{2}{y^2} - 2\frac{1 + y^2}{y} + 4V_1 - y^2(16V_1 + 1) \tag{35}$$

Now, we achieve the generalized form of wave function, which is corresponding to usual wave function $\hat{f}(x)$ as

$$\hat{f}(y,t) = \Delta f(y,t) = \frac{-1}{y}(1 - y^2)\left(1 + \partial_y\right)f(y,t), \quad y = sin\left(\frac{x}{2}\right)$$

$$\hat{f}(x,t) = \frac{cos^2\left(\frac{x}{2}\right)}{sin\left(\frac{x}{2}\right)}\left\{\left[1 + \frac{1}{2}V_1\sin(2x) - \frac{\beta - \alpha}{sin\left(\frac{x}{2}\right)} + \frac{\alpha + \beta + 3}{4}cot\left(\frac{x}{2}\right)\right]f(x,t)\right.$$
$$\left. + e^{-\frac{1}{2}V_1\left(1 + 2cos^2\left(\frac{x}{2}\right)\right)}\left\{\left[\frac{1 + cos\left(\frac{x}{2}\right)}{1 - cos\left(\frac{x}{2}\right)}\right]^{\frac{\beta - \alpha}{4}} sin\left(\frac{x}{2}\right)^{\frac{\alpha + \beta + 3}{2}} p'^{\alpha,\beta}_{n,m}(x)\right\}\right\} \tag{36}$$

5. Darboux Transformation and trigonometric sine-squared potential

The one-dimensional Schrödinger equation for the trigonometric sine-squared potential is given by

$$C''(y) = \frac{3y}{1 - y^2}C'(y) + \left(\frac{\omega}{1 - y^2} - \mu\right)C(y) \tag{37}$$

Thus, the trigonometric sine-squared potential equation (37) is

$$(1 - y^2)C_{yy} - 3yC_y - \left(\omega - \mu(1 - y^2)\right)C = 0 \tag{38}$$

By taking $F = (1 - y^2)$, $G = -3y$ and the potential, $V = \omega - \mu(1 - y^2)$, we can rewrite the above equation as,

$$F\partial_{yy} + G\partial_y - V = 0 \tag{39}$$

And

$$iC_t + \eta C = 0, \quad \eta = F\partial_{yy} + G\partial_y - V \tag{40}$$

In order to have same equations as (38) and (40) with different potentials, we have to write following equation,

$$i\hat{C}_t + \hat{\eta}C = 0, \quad \hat{\eta} = F\partial_{yy} + G\partial_y - \hat{V} \tag{41}$$

Where $V \neq \hat{V}$ and this imply $C \neq \hat{C}$. In order to obtain the modified potential $\hat{V}$ and corresponding wave function for equation (41), we introduce operator Δ, which is called Darboux transformation. The general form of such Durboux transformation will be

$$\Delta = A + B\partial_y \tag{42}$$

For simplicity we suppose *A* = *B*. By using the following property of Darboux transformation

$$\Delta(i\partial_t + \eta) = (i\partial_t + \hat{\eta}) \tag{43}$$

One can obtain the generalized form of wave function, which is corresponding to usual wave function *C* as

$$\hat{C}(y,t) = \Delta C(y,t) = \left(1 + \partial_y\right)C(y,t), \quad y = sin\frac{x}{a}$$

$$\hat{C}(x,t) = \left[1 - \frac{\alpha+\beta+1}{2}\left(\frac{tan\frac{x}{a}}{cos\frac{x}{a}}\right) + \frac{\beta-\alpha}{cos\frac{x}{a}}\right] C(x,t) + a\left[\frac{1+sin\frac{x}{a}}{1-sin\frac{x}{a}}\right]^{\left(\frac{\beta-\alpha}{4}\right)} cos\left(\frac{x}{a}\right)^{\left(\frac{\alpha+\beta-1}{2}\right)} P'^{\alpha,\beta}_{n,m}(x) \quad (44)$$

In order to obtain the parameter A, we need to use the equations (38) and (43) in following expression

$$\Delta\left(i\partial_t + F\partial_{yy} + G\partial_y - V\right)C = \left(i\partial_t + F\partial_{yy} + G\partial_y - \hat{V}\right)\Delta C$$

(45)

Making linear independence of C and its partial derivatives, we collect their respective coefficients and equal them to zero, so we can obtain A as

$$A = \alpha\sqrt{F} = \alpha\sqrt{1-y^2} \quad (46)$$

And the modified potential is given by

$$\hat{V} = a^2V - a^2E + 2tan^2\frac{x}{a} + 2sin\frac{x}{a}$$

(47)

Where, $C(y,t) = e^{-\frac{iEt}{\hbar}}C(y)$

6. Super-symmetry and Darboux Transformation

In what follows, we will prove that the formalism of supersymmetry for our generalized trigonometric Double-Cosine potential equation is equivalent to the Darboux transformation. So, here we introduce the following self- adjoint operator

$$(i\partial_t + \theta)^* = i\theta + \theta \quad (48)$$

Taking the operation of conjugation on Darboux transformation in Eqn. (21), we obtain

$$(i\theta_t + \eta)\Delta^* = \Delta^*(i\partial_t + \hat{\eta}) \quad (49)$$

Where the operator Δ^* ad joint to $\Delta = \frac{-1}{y}(1-y^2)\left(1+\partial_y\right)$ in double-cosine system is given by

$$\Delta^* = \frac{-1}{y}(1-y^2)\left(1-\partial_y\right) \quad (50)$$

Eqns. (29) and (30) can be rewritten by single matrix equation

$$\begin{bmatrix} i\partial_t + \eta & 0 \\ 0 & i\partial_t + \hat{\eta} \end{bmatrix}\begin{bmatrix} f \\ \hat{f} \end{bmatrix} = 0 \quad (51)$$

We assume that $H = diag(\eta, \hat{\eta})$ and $F = \left(f, \hat{f}\right)^T$, so the above equation can be written as

$$[i\partial_t + H]F = 0 \quad (52)$$

Two supercharge operator Q and Q^* are defined by following matrices

$$Q = \begin{bmatrix} 0 & 0 \\ \Delta & 0 \end{bmatrix}, \quad Q^* = \begin{bmatrix} 0 & \Delta^* \\ 0 & 0 \end{bmatrix}$$

(53)

Where Δ and Δ^* are the operator given by Eqns. (36) and (50), respectively. One can show that the Hamiltonian H satisfies the following expressions

$$\{Q,Q\} = \{Q^*,Q^*\} = 0$$
$$[Q, i\partial_t + H] = [i\partial_t + H, Q]$$
$$[Q^*, i\partial_t + H] = [i\partial_t + H, Q^*]$$

(54)

Considering the complementing relations of the super-symmetry algebra; the anti commutators are $\{Q,Q\}$ and $\{Q^*,Q^*\}$. We obtain the operators $R = Q^*Q$ and $\hat{R} = QQ^*$ and consider their relations with our Hamiltonian η and $\hat{\eta}$ So, one obtain the R and $\hat{R}$ as follow,

$$R = |\alpha|^2\left[F\left(1-\partial_{yy}\right) - (F)_y\left(\partial_y + 1\right)\right]$$

(55)

And

$$\hat{R} = |\alpha|^2\left[F\left(1-\partial_{yy}\right) - (F)_y\left(\partial_y+1\right) - \frac{1}{2}(F)_{yy} + \frac{F_y}{2F}\right]$$

(56)

Where, the index y will be derivative with respect to y. In order to have shape invariance and super-symmetric algebra we need to obtain, $\hat{R} - R$. If such value be constant or zero there is some super-symmetry partner for such systems. Otherwise we need to apply some condition in $\hat{R} - R$ to have constant value. So, we will arrive at following equation for the $\hat{R} - R$

$$\hat{R} - R = |\alpha|^2\left[1 + sin\left(\frac{x}{2}\right)\right] \quad (57)$$

By using the condition $\Psi(0) = \Psi(2\pi) = 0$ and, $x \in [0,2\pi]$ the value of $\hat{R} - R$ be zero or function of α, and we have super-symmetry for the Double-Cosine potential in case of α constant. So, in general we can say that there is shape invariance for usual and generalized potential under the above mentioned condition. The shape invariance for the potential is: $\hat{V} = V + constant$.

In second example, we consider sine-squared potential, so Δ and Δ^* will be

$$\Delta = \alpha\sqrt{1-y^2}\left(1+\partial_y\right) \quad (58)$$

And

$$\Delta^* = \alpha\sqrt{1-y^2}\left(1-\partial_y\right) \quad (59)$$

For the sine-squared potential also we consider information from previous section such as equations (51-54) and $R = Q^{*}Q$ and $\hat{R} = QQ^{*}$, one can obtain R and $\hat{R}$ as(55) and (56).
Otherwise, we need to apply some condition in $\hat{R} - R$ to have constant value. So, one can obtain the following equation for the$\hat{R} - R$

$$\hat{R} - R = |\alpha|^2 \left(1 - \frac{y}{1 - y^2}\right) \tag{60}$$

We mention here that if we want to super-symmetry algebra we need to have also the following commutation relation, and also anti-commutation relations between Q andQ^{+}.

$$\{Q, Q^{+}\} = H \ , \quad \{Q, Q\} = \{Q^{+}, Q^{+}\} = 0 \tag{61}$$

If we look at the equation (61), we need to apply the condition $\hat{R} - R$ be zero or constant, in the corresponding condition $\hat{R} - R$ be zero or function of α (α is constant). So, we have super-symmetry system, and it means that two potentials are satisfied by the shape invariance condition.

7. Conclusion

In this paper, the Double-Cosine potential equation was studied. The first-order Darboux transformation was applied to the corresponding equation. In order to relate super-symmetry and Darboux transformation, we discussed the supersymmetry algebra and its commutation and anti-commutation relations. It was shown that to satisfy such anti-commutation super-charges, the term $\hat{R} - R$ must be constant. Also, we applied this condition to $\hat{R} - R$ and showed that in the interval$[0,2\pi]$, α must be constant. This condition completely guarantees the relation between super-symmetry and Darboux transformation. This result plays an important role for any solvable, non-solvable and quasi-solvable systems.

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